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$\mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$

$\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$? rational numbers

$\downarrow$

to make this precise, we need equivalence classes

A binary relation R between two sets A and B is a subset $R \subseteq A \times B$.

$\Rightarrow$ $(a, b) \in R \Leftrightarrow aRb$

A function $F: A \to B$ is a relation such that if aFb and aFb' then $b = b'$.

$\Rightarrow$ $aFb \Leftrightarrow F(a) = b$

An equivalence relation on a set A is a relation $R \subseteq A \times A$ such that

1. R is reflexive: $aRa \ \forall a \in A$
2. R is symmetric: $aRb \Leftrightarrow bRa$
3. R is transitive: aRb and bRc imply aRc

$\Rightarrow$ $aRb \Leftrightarrow a \sim b$

Let A be a set and $\sim$ an equivalence relation on A.
The equivalence class of $a \in A$ is $[a] = \{b \in A \mid b \sim a\}$

The quotient set of the set A by the eq. relation $\sim$ is the set of eq. classes of A under $\sim$, i.e.

$$A/\sim \; = \{[a] \mid a \in A\}$$

Let $Q = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$.
Let $\sim$ be defined by $(a, b) \sim (c, d) \Leftrightarrow ad = bc$. $\left(\frac{a}{b} = \frac{c}{d}\right)$

$\mathbb{Q} = Q/\sim \; = \{[(0,1)], [(7,7)], [(-9,-18)], \ldots\} = \{0, \pm 1, \pm \frac{1}{2}, \ldots\}$ * In fact, $\mathbb{Q}$ is a field.

$[(0,1)] = \{(0,1), (0,2), (0,\pm 3), \ldots\}$

A field F is a set with two operations called addition and multiplication satisfying the following axioms:

- Addition: $\forall \ x, y, z \in F$
  1. closure: $x + y \in F$
  2. commutativity: $x + y = y + x$
  3. associativity: $x + (y + z) = (x + y) + z$
  4. additive identity: $\exists \ 0 \in F$
  5. additive inverses: $\exists \ -x \in F$ such that $x + (-x) = 0$
- Multiplication: $\forall \ x, y, z \in F$
  1. closure: $xy \in F$
  2. commutativity: $xy = yx$
  3. associativity: $x(yz) = (xy)z$
  4. multiplicative identity: $\exists \ 1 \in F$ s.t. $1x = x$
  5. multiplicative inverses: $\forall x \neq 0, \exists \ x^{-1}$ s.t. $x \cdot x^{-1} = 1$
- Distributive: $x(y + z) = xy + xz$

Remark: $\mathbb{Q}$ has holes.

$\Rightarrow$ Ex. $\sqrt{2} \notin \mathbb{Q}$

Pf. Suppose $\sqrt{2} \in \mathbb{Q}$. Then

$$\sqrt{2} = \frac{a}{b} \quad \text{(not both even)}$$

Then $2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2$. So a^2 is even. Thus, a is even, so a^2 is divisible by 4. Thus, 4 divides $2b^2$, so b^2 is even. Hence b^2 is even, which violates our initial condition. Thus, $\sqrt{2} \notin \mathbb{Q}$.