Exam: * $F \subseteq A \times B$

(1) $(a,b) \in F$, $(a,b') \in F$
$\Rightarrow b = b'$

(2) $\forall a \in A \ \exists b \in B$ s.t. $(a,b) \in F$

(a) $p_n \to p$, $p_n \to p' \Rightarrow p = p'$ (True)

Proof: Let $\varepsilon > 0$. Then $\exists N_1 \in \mathbb{N}$ such that $n > N_1 \Rightarrow d(p_n, p) < \varepsilon$.
But $p_n \to p'$ also. Thus $\exists N_2 \in \mathbb{N}$ such that $n > N_2 \Rightarrow d(p_n, p') < \varepsilon$.
Since $\varepsilon > 0$, we can write

$$d(p_n, p) < \varepsilon/2 \quad \text{and} \quad d(p_n, p') < \varepsilon/2.$$

Let $N = \max(N_1, N_2)$. Then, for $n > N$,

$$d(p, p') \leq d(p, p_n) + d(p_n, p')$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad (\text{for all } \varepsilon > 0).$$

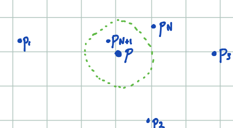
Thus $d(p, p') = 0$, so $p = p'$. ∎

(b) bounded $\longrightarrow$ converges (False)

$\longrightarrow$ counterexample!

(c) $\{p_n\}$ converges $\Rightarrow$ $\{p_n\}$ bounded (True)

Proof: Suppose $p_n \to p$. Note $1 > 0$. Thus, $\exists N \in \mathbb{N}$ such that
$n > N \Rightarrow d(p_n, p) < 1$.



Since n is finite, we let
$r > \max\{1, d(p, p_1), d(p, p_2), \dots, d(p, p_N)\}$

Then $p_i \in N_r(p)$ for all i. Thus Range $(\{p_n\}) \subseteq N_r(p)$,
so $\{p_n\}$ is bounded.

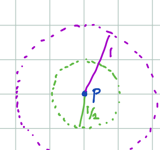
Remark: "all but finitely many" = "almost all"

(d) $p_n \to p \Rightarrow p$ is a limit point of the range $\{p_n\}$ (False)

Let $p_n = 5 \ \forall n \in \mathbb{N}$ $\longrightarrow$ counterexample

(e) $p \in E' \subseteq X \Rightarrow \exists \{p_n\}$ such that $p_n \to p$. (True)

Proof: Let $p \in E'$.



$\exists \ p_1 \neq p$, $p_1 \in E$, $p_1 \in N_1(p)$
$\exists \ p_2 \neq p$, $p_2 \in E$, $p_2 \in N_{1/2}(p)$

For $n \in \mathbb{N}$ $\exists$ $p_n \neq p$, $p_n \in E$, $p_n \in N_{1/n}(p)$.
These form a sequence $\{p_n\}$. We claim $p_n \to p$.

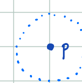
Let $\varepsilon > 0$. We must find $N \in \mathbb{N}$ such that $n > N \Rightarrow d(p_n, p) < \varepsilon$.

But, by the Archimedean property, $\exists\ n \in \mathbb{N}$ such that
$N\varepsilon > 1$.

Thus, $\frac{1}{N} < \varepsilon$. So, for $n > N$, $p_n \in N_{1/n}(p) \Rightarrow d(p_n, p) < 1/N < \varepsilon$. ∎

(f) $p_n \to p \Leftrightarrow$ every $N_r(p)$ contains almost all terms in $\{p_n\}$ (True)

Proof: ($\Rightarrow$) Suppose $p_n \to p$. Given $\varepsilon > 0$, $\exists\ N$ such that
$n > N \Rightarrow d(p_n, p) < \varepsilon$, i.e. $p_n \in N_\varepsilon(p)$.
At most $p_1, \dots, p_n$ outside $N_\varepsilon(p)$.



($\Leftarrow$) For any $\varepsilon > 0$, $N_\varepsilon(p)$ contains almost all $\{p_n\}$. So
$\exists\ N \in \mathbb{N}$ such that $p_n \in N_\varepsilon(p)$. Thus $p_n \to p$. ∎

Theorem (Limit Laws):

Let $\{s_n\}$ and $\{t_n\}$ be sequences in $\mathbb{C}$, and $s_n \to s$, $t_n \to t$.
Then the following hold:

(a) $\lim_{n\to\infty} (s_n + t_n) = s + t$

(b) $\lim_{n\to\infty} (c\, s_n) = cs$ for $c \in \mathbb{C}$.

(c) $\lim_{n\to\infty} (c + s_n) = c + s$ for $c \in \mathbb{C}$.

(d) $\lim_{n\to\infty} (s_n t_n) = st$

(e) if $s_n, s \neq 0$, then $\forall\, n \in \mathbb{N}$, then $\frac{1}{s_n} \to \frac{1}{s}$.

Proof: (a) Idea: $|(s_n + t_n) - (s+t)| \leq |s_n - s| + |t_n - t|$.

Let $\varepsilon > 0$. $s_n \to s \Rightarrow \exists\ N_1$ such that $n > N_1 \Rightarrow d(s_n, s) < \varepsilon/2$.
$t_n \to t \Rightarrow \exists\ N_2$ such that $n > N_2 \Rightarrow d(t_n, t) < \varepsilon/2$.

If $n > \max(N_1, N_2)$, then
$$d((s_n + t_n), (s+t)) = |s_n + t_n - s - t|$$
$$= |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(b) $|c s_n - cs| \leq |c|\,|s_n - s|$

(d) $|s_n t_n - st| = |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)|$

(e) $\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| < \frac{2}{|s|^2}|s_n - s|$ for $s_n \in N_{1/2|s|}(s)$

Def: The sequence $\{p_n\}$ is Cauchy if $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ such that
$n, m > N \Rightarrow d(p_n, p_m) < \varepsilon$.

Theorem: If $\{p_n\}$ is convergent, then $\{p_n\}$ is Cauchy.

Note: Not every Cauchy sequence is convergent!
Ex. Let p_n be the smallest $\frac{m}{n} > \pi$ in $\mathbb{Q}$ $\rightarrow$ does not converge in $\mathbb{Q}$.

Theorem: If X is compact, then every Cauchy sequence in X is convergent.
↳ such spaces are complete (Cauchy $\Leftrightarrow$ compact).