Let $\{p_n\}$ be a sequence in X. Let $\{n_i\}$ be a sequence in $\mathbb{N}$ such that

$$n_1 < n_2 < n_3 < \ldots$$

Then $\{p_{n_i}\}$ is a subsequence of $\{p_n\}$.

Ex. Let $\{p_n\} = \{1, \pi, \frac{1}{2}, \pi, \frac{1}{3}, \pi, \ldots\}$

$\{p_{2n}\} = \{\pi, \pi, \pi, \ldots\} \longrightarrow \pi$

$\{p_{2n+1}\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \longrightarrow 0$

Note: if $p_n \to p$, then $p_{n_k} \to p$ for any subsequence $\{p_{n_k}\}$ of $\{p_n\}$.

Does every sequence have a convergent subsequence?

<u>No.</u> $\{p_n\} = \{1, 2, 3, \ldots\} \to$ no convergent subsequence

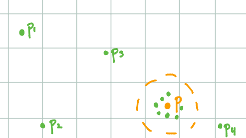
Lemma: In a compact metric space X, every sequence has a convergent subsequence converging to a point in X.

Proof: Let $\{p_n\}$ be a sequence in X. Let $R = \text{Range}(\{p_n\})$.

If R is a finite set, $\exists\ x \in R$ such that $p_n = x$ for infinitely many $n \in \mathbb{N}$. Call these terms $\{p_{n_k}\}$.

Then $p_{n_k} \to x$.

Otherwise, $R \subseteq X$ is infinite, so R has a limit point in X. Call it $p \in X$, $p \in R'$.



* reproving property (e).

Choose n_1 such that $d(p_{n_1}, p) < 1$ $(p_{n_1} \neq p)$.

For each i, choose $p_{n_i} \neq p$, $p_{n_i} \in N_{1/i}(p)$ and $n_i > n_{i-1}$. Then $p_{n_i} \to p$. $\blacksquare$

A metric space is <u>complete</u> if every Cauchy sequence is convergent.

Theorem: Compact metric spaces are complete.

Proof: Let $\{x_i\}$ be Cauchy in a compact metric space X.

Thus $\exists$ a subsequence $\{x_{n_k}\}$ converging to $x \in X$.

Fix $\varepsilon > 0$. Since $\{x_i\}$ is Cauchy, $\exists\ N \in \mathbb{N}$ such that $n, m > N \implies d(x_n, x_m) < \varepsilon/2$. Also, $x_{n_k} \to x$, so $\exists\ \tilde{N}$ such that $n_k > \tilde{N} \implies d(x, x_{n_k}) < \varepsilon/2$.

Let $\tilde{\tilde{N}} = \max(N, \tilde{N})$. Then

$$d(x, x_i) \leq d(x, x_{n_k}) + d(x_{n_k}, x_i)$$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

for $i, n_k \in \mathbb{N}$. ■

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. The upper limit of $\{s_k\}$ in $\mathbb{R} \cup \{\pm\infty\}$ is $\lim_{n\to\infty} \sup\{s_k \mid k > n\} = \lim\sup\{s_n\}$.

Similarly, the lower limit is $\lim_{n\to\infty} \inf\{s_k \mid k > n\} = \lim\inf\{s_k\}$.

Theorem: Let $\{s_n\}$ be real. Let $E \subseteq \mathbb{R} \cup \{\pm\infty\}$ be the set of all subsequential limits of $\{s_n\}$.

$$E = \{x \in \mathbb{R} \cup \{\pm\infty\} \mid s_{n_k} \to x \text{ for some } \{s_{n_k}\} \text{ of } \{s_n\}\}$$

Let $S^* = \sup E$, $S_* = \inf E$.

Then $\lim\sup\{s_n\} = S^*$, $\lim\sup\{s_n\} = S_*$.

HW (3.24): every metric space (X, d) has a completion (X^*, Δ).

Let $X^* = \{\text{Cauchy sequences in } X\}/\sim$.

$\{p_n\} \sim \{p_n'\}$ iff $\lim_{n\to\infty} d(p_n, p_n') = 0$.

- show eq. relation

for $\underset{\substack{\parallel \\ P}}{[\{p_n\}]}$, $\underset{\substack{\parallel \\ Q}}{[\{q_n\}]} \in X^*$

Let $\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n)$

- show well defined.

Ex: $X = \mathbb{Q}$, $X^* \underset{\text{isometrically isomorphic}}{\cong} \mathbb{R}$