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Let $\{s_n\}$ be a real sequence. We say

$\boxed{s_n \to +\infty}$,

$\{s_n\}$ diverges to $+\infty$ if, for each $x \in \mathbb{R}$,
$\exists\ N \in \mathbb{N}$ such that $n > N \Rightarrow s_n > x$.

Similar definition for $s_n \to -\infty$.

Let $\{s_n\}$ be a real sequence, and let E be the set of subsequential limits of $\{s_n\}$, i.e.

$$E = \left\{ x \in \mathbb{R} \cup \{\pm\infty\} \;\middle|\; s_{n_k} \to x \text{ for some } \{s_{n_k}\} \right\}.$$

Note: $E \neq \emptyset$ because it has a limit point if $\{s_n\}$ is bounded, and contains $+\infty$ or $-\infty$ if $\{s_n\}$ is unbounded.

$s^* = \sup E$ $\qquad\qquad$ $s_* = \inf E$

Theorem: (a) $s^*, s_* \in E$

(b) If $x > s^*$, then $\exists\ N \in \mathbb{N}$ such that $n > N \Rightarrow s_n < x$.

Also, s^* is the only such number.
Similarly for s_*.

Proof: (a) If $s^* = +\infty$, then E is not bounded above. Thus, $\exists\ \{s_{n_k}\}$ which diverges to $+\infty$. Thus, $+\infty \in E$, so $s^* \in E$.

If $s^* \in \mathbb{R}$, then E is bounded above, and at least one subsequential limit exists. E is closed (Theorem 3.7). Thus $\sup(E) \in \bar{E} = E$ (Theorem 2.28).

If $s^* = -\infty$, then every subsequence diverges to $-\infty$, so $s_n \to -\infty$. Thus $E = \{-\infty\}$, so $s^* \in E$.

Theorem: $s^* = \limsup s_n$ and $s_* = \liminf s_n$.

Proof: (sketch) Let $s_{n_k} \to t \in E$. Then
$\limsup s_{n_k} = t = \liminf s_{n_k}$.

But $\{s_{n_k}\} \subseteq \{s_n\}$ $\big(\text{range}(\{s_{n_k}\}) \subseteq \text{range}(\{s_n\})\big)$. So,
$\liminf s_n \leq \liminf s_{n_k} = t = \limsup s_{n_k} \leq \limsup s_n$

This holds $\forall\ t \in E$. Thus,
$\liminf s_n \leq \inf E \leq \sup E \leq \limsup s_n$

But $s^*, s_* \in E$. Thus, $\limsup s_n = s^*$, $\liminf s_n = s_*$.

* $\limsup s_n = \lim\limits_{n\to\infty} \sup \{ s_k \mid k > n\}$

$\limsup s_{n_k} = \lim\limits_{k\to\infty} \sup \{ s_{n_j} \mid j > k\}$

Theorem: $s_n \to s \iff \limsup s_n = \liminf s_n = s$ $(E = \{s\})$.

Ex Let $p > 0$.

(a) $\lim_{n\to\infty} \frac{1}{n^p} = 0$

(b) $\lim_{n\to\infty} p^{1/n} = 1$

(c) $\lim_{n\to\infty} n^{1/n} = 1$

(d) $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0 \quad \forall\, \alpha \in \mathbb{R}$

(e) if $|x| < 1$, $\lim_{n\to\infty} x^n = 0$

Series

$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$.

$1 - 1 + 1 - 1 + 1 - \ldots = ?$

$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots = \frac{1}{1-1/3} = 3/2$.

Let $\{a_n\}$ be a sequence in $\mathbb{R}$. For $i < j \in \mathbb{N}$,

$$\sum_{n=i}^{j} a_n = a_i + a_{i+1} + \ldots + a_j$$

The n^{th} partial sum of $\{a_n\}$ is

$$S_n = \sum_{k=1}^{n} a_k.$$

Note: $\{S_n\}$ is a sequence in $\mathbb{R}$.

This sequence $\{S_n\}$ is written $\sum_{n=1}^{\infty} a_n$ and is called an infinite series.

If $S_n \to s$, we write $\sum_{n=1}^{\infty} a_n = s$ and we say the series converges to s.