Series

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If $\{a_n\}$ is a real sequence, the sequence of partial sums is

$$S_n = \sum_{k=1}^{n} a_k$$

The sequence of partial sums $\{s_n\}$ is an infinite series, denoted $\sum_{n=1}^{\infty} a_n$.

$\sum_{n=1}^{\infty} a_n$ converges iff $\{s_n\}$ converges (as a sequence).

Ex. Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

Consider $\{s_n\}$, where $s_n = \sum_{k=1}^{n} \frac{1}{k}$.

$\mathbb{R}$ is complete, so $\{s_n\}$ converges iff it is Cauchy.

For $m < n$, $d(s_m, s_n) = s_n - s_m = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{m} \frac{1}{k} = \sum_{k=m+1}^{n} \frac{1}{k} = \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \geq \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n} = \frac{n-m}{n}$

Thus, $s_n - s_m \geq \frac{n-m}{n}$. Therefore, $s_{2n} - s_n \geq \frac{2n-n}{2n} = \frac{1}{2}$.

Let $\varepsilon < 1/2$. Then there is no $N \in \mathbb{N}$ such that $n, m > N \Rightarrow s_n - s_m < \varepsilon$.
Thus $\{s_n\}$ is not Cauchy, so $\{s_n\}$ is not convergent, i.e. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem: (Cauchy Criterion for Series)

$\sum_{n=1}^{\infty} a_n$ converges iff $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $m, n > N$ implies

$$\left| \sum_{k=n}^{m} a_k \right| < \varepsilon.$$

Corallary: (Divergence test)

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof: Let $\varepsilon > 0$. Then $\exists\ N \in \mathbb{N}$ such that $m, n > N$ implies
$$\left| \sum_{k=n}^{m} a_k \right| < \varepsilon.$$
Choose $m = n > N$. Then $|a_n| < \varepsilon$ for all $n > N$.
So $\lim_{n\to\infty} a_n = 0$.

Note: $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

Ex. $\sum_{n=1}^{\infty} (-1)^n$ diverges.

Note: converse is not true (harmonic series diverges).

Theorem: If $a_n \ge 0$, then $\sum_{n=1}^{\infty} a_n$ converges iff its sequence of partial sums $\{s_n\}$ is bounded.

Proof: Since $a_n \ge 0$, $\{s_n\}$ is monotonically increasing. Thus $\{s_n\}$ is bounded and monotonic. Hence $\{s_n\}$ converges.

If $\{s_n\}$ converges, then Range $(\{s_n\})$ is bounded. ■

Theorem: (Comparison Test)

(a) If Σc_n converges, and $|a_n| \le c_n$ for almost all $n \in \mathbb{N}$, then Σa_n converges.

(b) If Σd_n diverges to $+\infty$, and $a_n \ge d_n$ for almost all n, then Σa_n also diverges to $+\infty$.

Proof: (a) Let $\varepsilon > 0$. Since Σc_n converges, it is Cauchy. So $\exists\ N \in \mathbb{N}$ such that $m \ge n > N$ implies

$$\sum_{k=n}^{m} c_k \le \left| \sum_{k=n}^{m} c_k \right| < \varepsilon.$$

Thus, $$\left| \sum_{k=n}^{m} a_k \right| \le \underbrace{\sum_{k=n}^{m} |a_k| \le \sum_{k=n}^{m} c_k}_{} < \varepsilon.$$ holds for almost all n, just pick $n \gg N$.

So Σa_k satisfies the Cauchy criterion and thus converges.

(b) Contrapositive of (a). Also boundedness of partial sums.

Theorem: (Geometric Series)

If $|x| < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

If $|x| \ge 1$, then $\sum_{n=0}^{\infty} x^n$ diverges.

Proof: If $x \ne 1$, then

$$S_n = 1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}.$$

Thus, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x}$ converges if $|x| < 1$ and diverges if $|x| > 1$.

If $|x| = 1$, then Σx^n diverges by the divergence test.

Ex: $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

$$S_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \le \overbrace{1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}}^{\text{geometric series}} < 3.$$

Since $\{S_n\}$ is bounded and monotonic, $\Sigma \frac{1}{n!}$ converges.

Def: $\sum_{n=0}^{\infty} \frac{1}{n!} = e$.

Remark: $$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots = \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right) \\ &\le \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) \\ &= \frac{1}{(n+1)!}\left(\frac{1}{1-\frac{1}{n+1}}\right) = \frac{1}{(n+1)!}\left(\frac{1}{\frac{n}{n+1}}\right) = \frac{1}{n!\cdot n} \end{aligned}$$

Remark: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$