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Theorem: Let $\{a_n\}$ be non-negative and monotonically decreasing.

$$a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0.$$

Then $\sum a_n$ converges iff $\sum 2^k a_{2^k}$ converges.

Proof: We compare partial sums. Let

$$s_n = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \dots + a_n$$
$$t_k = a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \dots + (a_{2^k} + \dots + a_{2^k})$$

If $n \le 2^k$, then $s_n \le t_k$. So, if t_k converges, then s_n diverges $\left(\lim_{n\to\infty} n \le \lim_{k\to\infty} 2^k\right)$.

Otherwise, if $n > 2^k$, then $2s_n > t_k$.

$$2a_1 + 2a_2 + 2(a_3 + a_4) + \dots$$
$$a_1 + (a_2 + a_2) + 4a_4 + \dots$$

* Multiplying by a constant does not change convergence.

So, if s_n converges, t_k converges. ■

Theorem: $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Proof: If $p \le 0$, then $\frac{1}{n^p} \ge 1$. So $\lim_{n\to\infty} \frac{1}{n^p} \ne 0$, hence the series diverges by the divergence test.

If $p > 0$, then $\left\{\frac{1}{n^p}\right\}$ is non-negative and monotonically decreasing. By the previous theorem, $\sum \frac{1}{n^p}$ converges iff $\sum 2^k \frac{1}{(2^k)^p}$ converges.

$\sum 2^k \frac{1}{(2^k)^p} = \sum 2^{k(1-p)}$. This is geometric, with ratio 2^{1-p}. Hence it converges iff $2^{1-p} < 1$, which occurs iff $1 - p < 0 \Leftrightarrow 1 < p$. ■

Theorem: (Root Test) Given $\sum a_n$, let $\alpha = \limsup \sqrt[n]{|a_n|}$.

(a) $\alpha < 1 \Rightarrow \sum a_n$ converges
(b) $\alpha > 1 \Rightarrow \sum a_n$ diverges
(c) $\alpha = 1 \Rightarrow$ NO INFO

Proof: (a) If $\alpha < 1$, choose $\alpha < \beta < 1$.
Then $\exists\ N \in \mathbb{N}$ such that $n > N \Rightarrow \sqrt[n]{|a_n|} < \beta$.
Hence $|a_n| < \beta^n$ for $n > N$. Since $0 < \beta < 1$, $\sum \beta^n$ converges.
Thus $\sum a_n$ converges by comparison.

(b) If $\alpha > 1$, since $\alpha = \limsup \sqrt[n]{|a_n|}$, $\exists$ a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} \sqrt[n_k]{|a_{n_k}|} = \alpha$. Choose $\varepsilon < \alpha - 1$. Then $\exists\ N \in \mathbb{N}$ such that $n_k > N \Rightarrow \sqrt[n_k]{|a_{n_k}|} \in N_\varepsilon(\alpha)$. Therefore there are infinitely many terms of $\{a_n\}$ such that $|a_n| > 1$. So $\sum a_n$ diverges by the divergence test.

(c) Consider $\underset{\text{diverges}}{\sum \frac{1}{n}}$, $\underset{\text{converges}}{\sum \frac{1}{n^2}}$ with $\alpha = 1$. $\left(n^{1/n} \to 1\right)$

Theorem: (Ratio Test) Given $\sum a_n$, let $R = \limsup \left|\frac{a_{n+1}}{a_n}\right|$
$r = \liminf \left|\frac{a_{n+1}}{a_n}\right|$.

(a) $R < 1 \Rightarrow \sum a_n$ converges
(b) $r > 1 \Rightarrow \sum a_n$ diverges
(c) If $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for almost all $n \in \mathbb{N}$, then $\sum a_n$ diverges
(d) no info if $r = 1$ or $R = 1$ (same counter example)

A power series is of the form

$$\sum_{n=0}^{\infty} c_n z^n \quad \text{for} \quad c_n \in \mathbb{C}.$$

Theorem: Let $\alpha = \limsup \sqrt[n]{|c_n|}$, $R = 1/\alpha$. Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$. (Proof by root test.)

R is called the radius of convergence.

Ex: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ if $|x| < 1$.

$\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Theorem: If $\sum a_n$ converges but not absolutely, there is a rearrangement with any $\limsup a_n$ and $\liminf a_n$.

Theorem: (Alternating Series) If $a_1 \geq a_2 \geq \ldots \geq 0$, $\lim_{n \to \infty} a_n = 0$, then $\sum (-1)^n a_n$ converges.

Ex: $\sum (-1)^n \frac{1}{n}$ converges.