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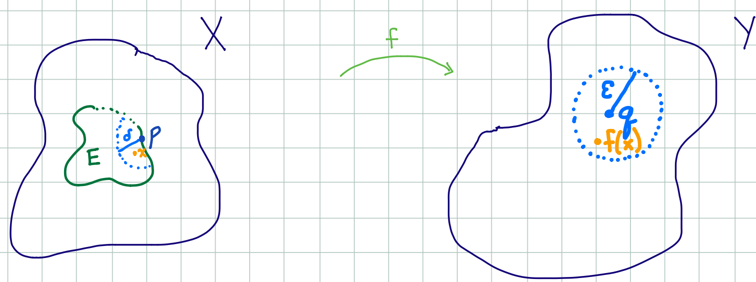
3. 31. 14

Let X and Y be metric spaces, $f: X \to Y$ be a function, $E \subseteq X$, $p \in E'$.
We say the limit as x approches p in E of $f(x)$ is q, denoted

$$\boxed{\lim_{x\to p} f(x) = q} \quad \text{or} \quad \boxed{f(x) \to q \text{ as } x \to p}$$

if $\boxed{\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } x \in E,\ 0 < d_X(x,p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon.}$

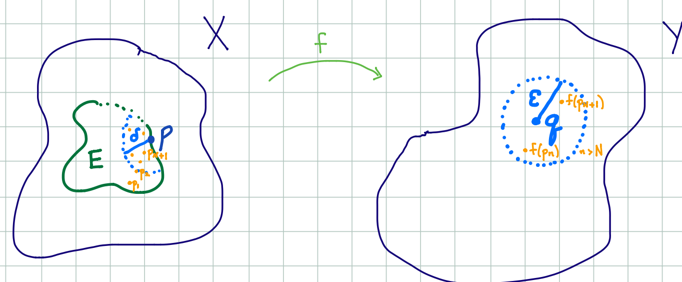
Remark: f need not be defined at $x = p$ (we only care about $d_X(x,p) > 0 \Rightarrow x \neq p$).



Theorem: Let X, Y, f, p, E be as above. Then $\lim_{x\to p} f(x) = q$ iff $\lim_{n\to\infty} \underbrace{f(p_n)}^{\text{sequence in } Y} = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$ and $p_n \to p$.

Proof: $(\Rightarrow)$ Let $p_n \to p$, $p_n \neq p$ $\forall\, n \in \mathbb{N}$. We assume $f(x) \to q$ as $x \to p$.
We must show that $\lim_{n\to\infty} f(p_n) = q$.

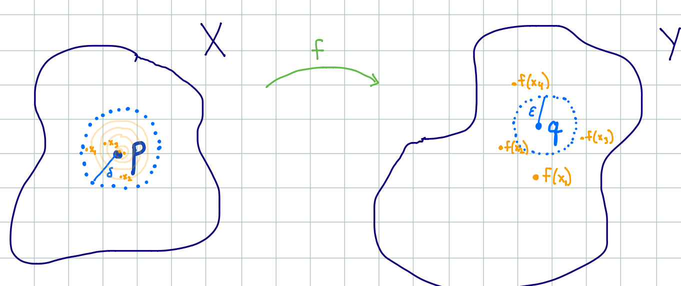
Let $\varepsilon > 0$. $\exists\, \delta > 0$ such that
$x \in E$, $0 < d_X(x,p) < \delta \Rightarrow$
$d_Y(f(x), q) < \varepsilon$.



But $p_n \to p$. So $\exists\, N \in \mathbb{N}$ such that
$n > N \Rightarrow d_X(p_n, p) < \delta$. Thus, $n > N$
$\Rightarrow 0 < d_X(p_n, p) < \delta \Rightarrow d_Y(f(p_n), q) < \varepsilon$.
Therefore, $\lim_{n\to\infty} f(p_n) = q$.

$(\Leftarrow)$ Contrapositive.

Suppose $\lim_{x\to p} f(x) \neq q$. Then $\exists$ some $\varepsilon > 0$ such that $\forall\, \delta > 0$,
$0 < d_X(x,p) < \delta \not\Rightarrow d_Y(f(x), q) < \varepsilon$. Thus $\exists\, x \in E$ such that
$0 < d_X(x,p) < \delta$ and $d_Y(f(x), q) \geq \varepsilon$.



Let $\delta_n = \frac{1}{n}$. Then choose $x_n \in E$ such that

$$0 < d_X(x_n, p) < \frac{1}{n} \quad \text{and} \quad d_Y(f(x_n), q)) \geq \varepsilon.$$

Then $x_n \to p$, $x_n \neq p$. But $\lim_{n\to\infty} f(x_n) \neq q$.

Remarks: • $\lim_{x\to p} f(x)$ is unique.

• limit laws apply!

Let $p \in E \subseteq X$, $f: E \to Y$. Then f is continuous at $p \in E$ if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that $d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$.

If f is continuous $\forall\, p \in E$, we say f is continuous on E.

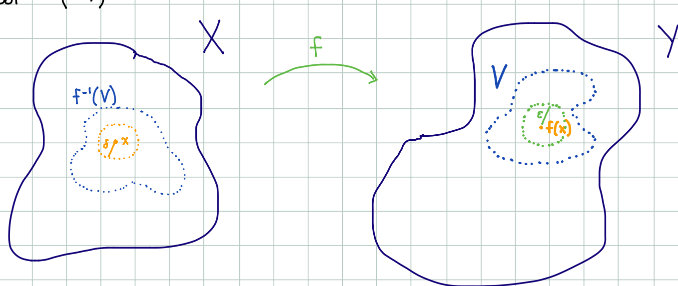
Theorem: Let $p \in E \cap E'$. Then f is continuous at p iff $\boxed{\lim_{x\to p} f(x) = f(p).}$

Proof: Let $f(p) = q$ in limit definition.

Remark: f is continuous on E iff $\forall$ convergent $\{x_n\} \in E$, $\boxed{\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).}$

Theorem: $f: X \to Y$ is continuous iff for each open $V \subseteq Y$, $f^{-1}(V) \subseteq X$ is open in X.

Proof: $(\Rightarrow)$



Let $V \subseteq Y$ be open. We wish to show that $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X.
Let $x \in f^{-1}(V)$. We show x is an interior point of $f^{-1}(V)$.

$x \in f^{-1}(V) \Leftrightarrow f(x) \in V$. $V \subseteq Y$ is open. So $f(x)$ is interior to V. Thus $\exists\, \varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq V$. Since f is continuous, $\exists\, \delta > 0$ such that $d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$. So $p \in B_\delta(x) \Rightarrow f(p) \in V$ $(p \in f^{-1}(V))$.
So x is interior.