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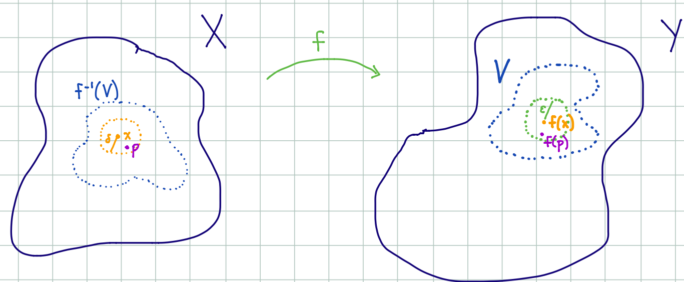
4.2.14

Theorem: $f: X \to Y$ is continuous iff for each open $V \subseteq Y$, $f^{-1}(V) \subseteq X$ is open in X.
($\Rightarrow$ continuous everywhere on X)

Proof: ($\Rightarrow$)

Suppose $f: X \to Y$ is continuous.

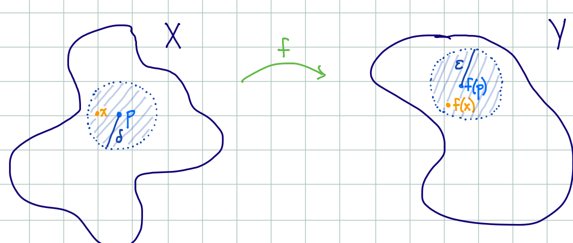
Let $V \subseteq Y$ be open. We wish to show that $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X. Let $x \in f^{-1}(V)$. We show x is an interior point of $f^{-1}(V)$. (If no such x exists, i.e. $f^{-1}(V) = \emptyset$, we are done.)



$x \in f^{-1}(V) \Leftrightarrow f(x) \in V$. $V \subseteq Y$ is open. So $f(x)$ is interior to V. Thus $\exists\ \varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq V$. Since f is continuous, $\exists\ \delta > 0$ such that $d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$. So $p \in B_\delta(x) \Rightarrow f(p) \in V \Leftrightarrow p \in f^{-1}(V)$. Therefore $B_\delta(x) \subseteq f^{-1}(V)$, so x is interior. Hence $f^{-1}(V)$ is open

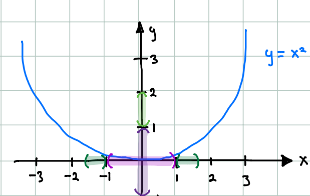
($\Leftarrow$)

Assume $f^{-1}(V)$ is open for all open $V \subseteq Y$. Let $p \in X$. We show f is continuous at p.



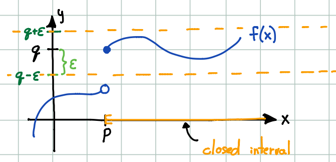
Let $\varepsilon > 0$. Let $V = N_\varepsilon(p)$. V is open, hence $f^{-1}(V) \subseteq X$ is open. Thus $p \in f^{-1}(V)$ is an interior point, so $\exists\ \delta > 0$ s.t. $N_\delta(p) \subseteq f^{-1}(V)$. So $d_X(x,p) < \delta \Rightarrow x \in f^{-1}(V)$ i.e. $f(x) \in N_\varepsilon(f(p))$ i.e. $d_Y(f(x), f(p)) < \varepsilon$. Hence f is continuous. ∎

Ex: $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$



$f^{-1}((1,2)) = (1, \sqrt{2}) \cup (-\sqrt{2}, -1)$
$f^{-1}((0, +\infty)) = \mathbb{R} \setminus 0$
$f^{-1}((-3,-2)) = \emptyset$
$f^{-1}((-1,1)) = (-1,1)$

Ex:



$f^{-1}((q-\varepsilon, q+\varepsilon))$ NOT OPEN

Cor: $f: X \to Y$ is continuous iff $K \subseteq Y$ closed $\Rightarrow f^{-1}(K)$ is closed in X.

Lemma: First, note that

$$(f^{-1}(K))^c = \{x \in X \mid x \notin f^{-1}(K)\} = \{x \in X \mid f(x) \notin K\} = \{x \in X \mid f(x) \in K^c\} = f^{-1}(K^c).$$

Proof: $K \subseteq Y$ closed $\Leftrightarrow K^c$ is open $\Rightarrow f^{-1}(K^c)$ is open $\Leftrightarrow (f^{-1}(K))^c$ is open $\Leftrightarrow f^{-1}(K)$ is closed. ∎

Theorem: Continuous maps preserve compactness. Let $F: X \to Y$ be continuous. If X is compact, then $f(X)$ is also compact.

Proof: Let $\{U_\alpha\}$ be an open cover of $f(X)$,

$$f(X) = \bigcup_\alpha U_\alpha. \qquad (U_\alpha \text{ open in } Y)$$

$f^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha f^{-1}(U_\alpha)$. Each $f^{-1}(U_\alpha)$ is open, so $\{f^{-1}(U_\alpha)\}$ form an open cover of X. Since X is compact, $\exists$ a finite subcover $X = f^{-1}(U_{\alpha_1}) \cup \dots \cup f^{-1}(U_{\alpha_n})$.

Thus $f(X) = f(f^{-1}(U_{\alpha_1})) \cup \dots \cup f(f^{-1}(U_{\alpha_n}))$.
$= U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$
because f is surjective onto $f(X)$.

Hence $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a finite subcover of $f(X)$. ∎

Prop: $f: A \to B$ is surjective $\Rightarrow f(f^{-1}(B)) = B$.

Proof: $f^{-1}(B) = \{a \in A \mid f(a) \in B\} = A$.

$f(f^{-1}(B)) = f(A)$. But f is surjective, i.e. $\forall\, b \in B$ $\exists\, a \in A$ s.t. $f(a) = b$. Thus $F(A) = B$, so $f(f^{-1}(B)) = B$.

Ex. $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = 2$ (compact)

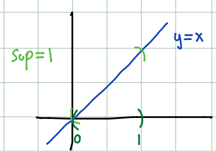
$$f(f^{-1}(\mathbb{R})) = 2 \neq \mathbb{R}$$

Cor: $f: X \to \mathbb{R}^k$ is continuous and X is compact $\Rightarrow f(x)$ is closed and bounded.

Cor: Let $f: X \to \mathbb{R}$ be continuous, X compact. Then f achieves its minimum and maximum values.

Pf: $f(X)$ is closed and bounded. Thus, $\sup\{f(x) \mid x \in X\}$ and $\inf\{f(x) \mid x \in X\}$ are in $f(X)$. I.e., $\exists\ x_1, x_2 \in X$ such that
$$f(x_1) = \sup\{f(x) \mid x \in X\}$$
$$f(x_2) = \inf\{f(x) \mid x \in X\}.$$

Ex.



no $x \in (0,1)$ st. $f(x) = 1$!

Q: Suppose $f: X \to Y$ is a bijection. If f is continuous, is $f^{-1}: Y \to X$ continuous?

A: No. $f: [0, 2\pi) \to S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
$$\theta \to (\cos\theta, \sin\theta)$$