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4.7.14

Theorem: Let $f: X \to Y$ be a continuous bijection. If X is compact, then $f^{-1}: Y \to X$ is continuous.

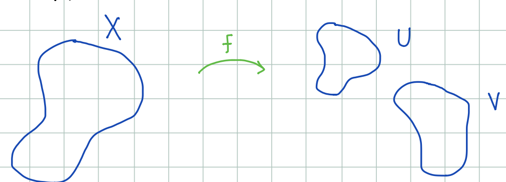
Proof: To show that $f^{-1}: Y \to X$ is continuous, we must show $(f^{-1})^{-1}(U)$ is open $\forall$ open $U \subseteq X$. But f is a bijection, hence $(f^{-1})^{-1} = f$. So we study $f(U)$. $U \subseteq X$ is open. Thus $X \setminus U$ is closed. $X \setminus U \subseteq X$ is a closed subset of the compact space X, hence $X \setminus U$ is compact. So $f(X \setminus U)$ is compact, as f is continuous. Hence $f(X \setminus U)$ is closed in Y, so $Y \setminus f(U) = f(X \setminus U)$ is closed. Thus $f(U)$ is open, so f^{-1} is continuous. ∎

Note: If $f: X \to Y$ is a bijection, f and f^{-1} are both continuous, f is called a <u>homeomorphism</u>.

Theorem: Continuous maps preserve connectedness. Let $f: X \to Y$ be continuous. If X is connected, then $f(X)$ is connected.

Proof: (Contrapositive)

Suppose $f(X)$ is not connected. We show X is not connected.



Then $\exists$ $U, V \subseteq Y$ open such that $U \cup V = f(X)$ and $U \cap V = \emptyset$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open. $f(X) = U \cup V$, thus $X = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. Also, $x \in f^{-1}(U) \cup f^{-1}(V)$ implies $f(x) \in U \cap V = \emptyset$. Thus $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence X is not connected. ∎

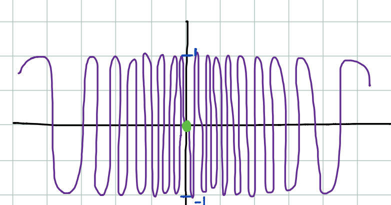
Theorem: (Intermediate Value Theorem)

Let $f: [a,b] \to \mathbb{R}$ be continuous. If $c \in \mathbb{R}$ is between $f(a)$ and $f(b)$, then $\exists$ $x \in (a,b)$ such that $f(x) = c$.

Proof: WLOG, assume $f(a) < c < f(b)$. If no such x exists, then $(f(a), c) \cup (c, f(b))$ disconnect $f((a,b))$. But (a,b) is connected. Hence $f((a,b))$ is connected by the previous theorem. ∎

Note: Converse of the IVT is false.

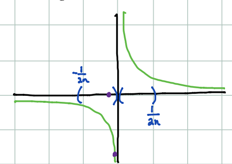
Counterexample: $f(x) \begin{cases} 0, & x = 0 \\ \sin\left(\frac{1}{x}\right), & x \neq 0 \end{cases}$



$f: X \to Y$ is <u>uniformly continuous</u> if, $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that $d_X(p, x) < \delta \Rightarrow d_Y(f(p), f(x)) < \varepsilon$.

Remark: same δ must work for all $p \in X$. If f is just continuous, δ may depend on $p \in X$.

Ex: <u>NOT</u> uniformly continuous



$f(x): \mathbb{R} \setminus \{0\} \to \mathbb{R}$
$f(x) = \frac{1}{x}$

$f(x)$ is continuous, but not uniformly.

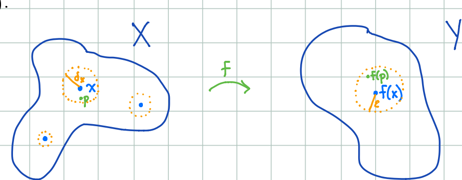
Let $\varepsilon > 0$. Choose any $\delta > 0$, and $p \in (-\frac{1}{2n}, 0)$, $q \in (0, \frac{1}{2n})$.
Thus $d(p,q) \le \frac{1}{2n} - (-\frac{1}{2n}) = \frac{1}{n}$. But $d(f(p), f(q)) = \frac{1}{q} - \frac{1}{p} > 4n$.
So $\exists\, n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$, and $\exists\, n' \in \mathbb{N}$ such that $4n' > \varepsilon$.
Choose $n^* = \max(n, n')$. Then $d(p,q) < \delta$ but $d(f(p), f(q)) > \varepsilon$.

Theorem: If f is uniformly continuous, then f is continuous.

Proof: Trivial. Try it.

Theorem: Let $f: X \to Y$ be continuous. If X is compact, then f is uniformly continuous.

Proof: Let $\varepsilon > 0$. We wish to find $\delta > 0$ that works for all $p \in X$. Each $x \in X$ has a δ_x such that $d_X(x,p) < \delta_x \Rightarrow d_Y(f(x), f(p)) < \varepsilon$ (by continuity at x).



Can we find δ such that $d(p,q) < \delta \Rightarrow p, q \in N_{\delta_x}(x)$ for some $x \in X$? Then $d(f(p), f(q)) \le d(f(p), f(x)) + d(f(x), f(q)) < 2\varepsilon$.

Lemma: (LeBesgue's Covering Lemma)

If $\{U_\alpha\}$ is an open cover of a compact space X, then $\exists\, \delta > 0$ such that $\forall\, x \in X$, $N_\delta(x) \subseteq U_\alpha$ for some α. (δ is the LeBesgue number of $\{U_\alpha\}$.)

Proof: Let $\{U_\alpha\}$ be an open cover. Suppose $\nexists$ such a δ. Let $\delta_n = \frac{1}{n}$. Then δ_n is not a LeBesgue number. But $\exists\, x_n, p_n \in X$ such that $d(x_n, p_n) < \delta_n$, $\nexists\, \alpha$ such that $x_n, p_n \in U_\alpha$.

I.e., $\exists\ x_n \in X$ such that $N_{\delta_n}(x_n) \nsubseteq U_\alpha$ for any α. $\{x_n\}$ is a subsequence in X, so $\exists\ x_{n_k} \to x$ (convergent subsequence). $\{U_\alpha\}$ is an open cover of X, so $\exists\ \alpha_0$ such that $x \in U_{\alpha_0}$. So $\exists\ r > 0$ such that $N_r(x) \subseteq U_{\alpha_0}$ ....