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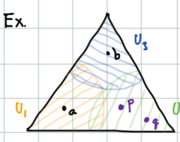
4.9.11

Lemma: (LeBesgue's Covering Lemma)

If $\{U_\alpha\}$ is an open cover of a compact space X, then $\exists\ \delta > 0$ such that $\forall\ x \in X$, $N_\delta(x) \subseteq U_\alpha$ for some α. (δ is the LeBesgue number of $\{U_\alpha\}$.)

Ex.

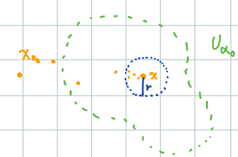


$a, b \notin U_i$ for $i = 1,2,3$

$p, q \in U_2$

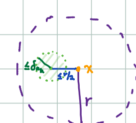
Proof: Let $\{U_\alpha\}$ be an open cover. Suppose $\nexists$ such a δ. Let $\delta_n = \frac{1}{n}$. Then δ_n is not a LeBesgue number. Thus $\exists\ x_n, p_n \in X$ such that $d(x_n, p_n) < \delta_n$ but $x_n, p_n \notin U_\alpha$ for any α.

I.e., $\exists\ x_n \in X$ such that $N_{\delta_n}(x_n) \nsubseteq U_\alpha$ for any α. $\{x_n\}$ is a sequence in X, and X is compact. Thus $\exists$ a convergent subsequence $\{x_{n_k}\}$, i.e. $\exists\ x \in X$ with $x_{n_k} \to x$.



Since $\{U_\alpha\}$ is an open cover of X, $\exists\ \alpha_0$ such that $x \in U_{\alpha_0}$. But U_{α_0} is open, so x is interior to U_{α_0}. Thus $\exists\ r > 0$ such that $N_r(x) \subseteq U_{\alpha_0}$. $x_{n_k} \to x$ implies $\exists\ N_1 \in \mathbb{N}$ such that $n_k > N_1$ implies $d(x_{n_k}, x) < r/2$. Also, $\exists\ N_2$ such that $n > N_2$ implies $\frac{1}{n} < r/2$ (by the Archimedian property).

Thus, for $n_k > N_1, N_2$, let $p \in N_{\delta_{n_k}}(x_{n_k})$. So



$$\begin{aligned} d(p,x) \le d(p, x_{n_k}) + d(x_{n_k}, x) &< \delta_{n_k} + r/2 \\ &< r/2 + r/2 = r. \end{aligned}$$

Therefore, $N_{\delta_{n_k}}(x_{n_k}) \subseteq N_r(x) \subseteq U_{\alpha_0}$. This is a contradiction, so our assumption was false, and every open cover of X must have a LeBesgue number. ■

Theorem: $f: X \to Y$ continuous and compact $\Rightarrow f$ uniformly continuous.
(Proved this last time.)

Discontinuity

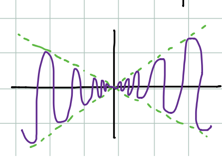
Let $f: (a,b) \to \mathbb{R}$. For all $\{t_n\}$ in (a,x) with $a < x < b$, and $t_n \to x$. If $f(t_n) \to q$, we write $\boxed{f(x^-) = q} \Leftrightarrow \lim_{t \to x} f(t) = q$.
Similarly, we have $\boxed{f(x^+) = \lim_{t \to x^+} f(t)}$.

If f is not continuous at x, but $f(x^+)$ and $f(x^-)$ exist, we say f has a discontinuity of the first kind, or a <u>simple discontinuity</u>.

Otherwise, it is a discontinuity of the <u>second kind</u>.

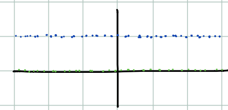
Ex. $f(x) = \begin{cases} 0, & x \le 0 \\ \sin(\frac{1}{x}) & x > 0 \end{cases}$ <u>2nd</u> (at $x=0$)

$f(x) = \begin{cases} 0, & x = 0 \\ x\sin(\frac{1}{x}), & x = 0 \end{cases}$



<u>continuous!</u>

$$f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$$



and everywhere!

$$f(x) = \begin{cases} \frac{1}{q}, & x = p/q \quad (\text{reduced in } \mathbb{Q}) \\ 0 & \text{otherwise} \end{cases}$$

This function is continuous at all the irrationals, and has simple discontinuities on $\mathbb{Q}$.

Sequences and Series of Functions

Let $f_n : X \to Y$ be a function for each $n \in \mathbb{N}$, then $\{f_n\}$ is a sequence of functions. If $\{f_n(a)\}$ converges in Y for each $a \in X$, we say $\{f_n\}$ converges pointwise to $f : X \to Y$ given by

$$f(x) = \lim_{n \to \infty} f_n(x).$$

If $\sum_{n=1}^{\infty} f_n(x)$ converges for each $x \in X$, then the series converges pointwise to

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Q: If f_n is continuous $\forall\, n \in \mathbb{N}$, if $f_n \to f$, is f continuous?
Is $\sum f_n$ continuous?

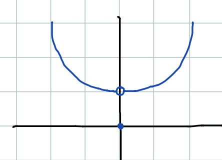
Ex. $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$

$\sum f_n = f$. Let $a \in \mathbb{R}$. $\sum_{n=0}^{\infty} \frac{a^2}{(1+a^2)^n}$ is geometric, so it converges.

If $x = 0$, $f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = 0. \to f(0) = 0.$

If $x \neq 0$, $f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = \frac{x^2}{1 - \frac{1}{1+x^2}} = 1 + x^2.$

$d(0, 1+x^2) > 1$ for all $x \neq 0$.
So it is discontinuous at 0!



simple discontinuity!