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Ex. $f_m(x) = \lim_{n\to\infty} (\cos(m!\pi x))^{2n}$

$m!x \in \mathbb{Z} \Rightarrow |\cos(m!\pi x)| = 1$
$\Rightarrow f_m(x) = 1$

Otherwise, $|\cos(m!\pi x)| < 1 \Rightarrow f_m(x) = 0$.

So, if $x \notin \mathbb{Q}$, then $f_m(x) = 0$.

$x \in \mathbb{Q} \Rightarrow x = \frac{p}{q}$, $\exists\ m \in \mathbb{N}$ such that $m!\frac{p}{q} \in \mathbb{Z}$ $(m \geq q)$.

Let $f(x) = \lim_{m\to\infty} f_m(x)$. Then

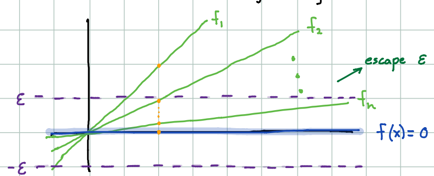
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

- $f(x)$ has discontinuities of the second kind at all $x \in \mathbb{R}$.

$\{f_n\}$ converges uniformly on X to a function $f: X \to Y$ if,
$\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $n > N \Rightarrow d(f_n(x), f(x)) < \varepsilon$ for all $x \in X$.

- one N works $\forall\ x$

Ex. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{x}{n}$.



Let $a \in \mathbb{R}$. $\lim_{n\to\infty} f_n(a) = \lim_{n\to\infty} \frac{a}{n} = 0$.

Thus $f_n \to f$ pointwise.

Let $\varepsilon = 1$. $|f_n(x) - f(x)| = |\frac{x}{n} - 0| = |\frac{x}{n}|$.
Can we find $N \in \mathbb{N}$ such that $|\frac{x}{n}| < 1$ $\forall\ x \in \mathbb{R}$, $n > N$?
$\to \forall\ x > n$, $|\frac{x}{n}| > 1$. So f_n does not converge uniformly to f.

Theorem: If $f_n \to f$ uniformly on X, then $f_n \to f$ pointwise on X. ✔
(Converse false by previous example.)

$\sum f_n$ converges uniformly if the sequence of partial sums $\{s_k\}$
$$s_k = \sum_{n=1}^{k} f_n$$
converges uniformly.

We set $f(x) = \lim_{n\to\infty} f_n(x)$. Recall that f is continuous at $x = a$ iff $\lim_{x\to a} f(x) = f(a)$.
Thus, f continuous at $x = a \Rightarrow \lim_{x\to a} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to a} f_n(x)$ (if each f_n is continuous!).

Note: Dangerous to exchange limits!

Ex. $S_{m,n} = \frac{m}{m+n}$. Fix n. Then $\lim_{m\to\infty} \frac{m}{m+n} = 1$. Fix m. Then $\lim_{n\to\infty} \frac{m}{m+n} = 0$.
So $\lim_{m\to\infty} \lim_{n\to\infty} S_{m,n} \neq \lim_{n\to\infty} \lim_{m\to\infty} S_{m,n}$.

Theorem: The uniform limit of a sequence of continuous functions is continuous.
Let $E \subseteq \mathbb{R}$, $f_i : E \to \mathbb{R}$, $f_n \to f$ uniformly. If f_n is continuous at $x_0 \in E$ for all $n \in \mathbb{N}$, then f is also continuous at x_0.

Proof: We must show $\lim_{x \to x_0} f(x) = f(x_0)$, i.e. $\lim_{x\to x_0} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x \to x_0} f_n(x)$.

Observe that $f(x) - f(x_0) = f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)$.

Thus, $|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$.

Let $\varepsilon > 0$. Then $\exists$ $N \in \mathbb{N}$ such that $n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon/3$ for all $x \in E$ (because $f_n \to f$ uniformly on E). Also, $\exists$ $\delta > 0$ such that $|f_{N+1}(x) - f_{N+1}(x_0)| < \varepsilon/3$ for all $x \in E$ such that $|x - x_0| < \delta$ (by continuity of f_{N+1}). So, $n = N+1$, $x \in E$, $|x - x_0| < \delta$ implies

$$|f(x) - f(x_0)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus, $\forall$ $\varepsilon > 0$, $\exists$ $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.
Hence f is continuous at x_0. ■

A function $f : [a,b] \to \mathbb{R}$ is <u>differentiable at $x \in [a,b]$</u> if $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$.

If f is continuous on $[a,b]$, is f differentiable on $[a,b]$?
→ <u>No</u>. ($f(x) = |x|$)

If f is differentiable on $[a,b]$, is f' continuous on $[a,b]$?
→ <u>No</u>. ($f(x) = x^{4/3} \sin(\frac{1}{x})$).