An order on a set S is a relation, denoted $<$, such that: Hannah Rose

1. trichotomy: $\forall\, x, y \in S$, one and only one of the following hold:
   $x<y$ or $x=y$ or $y<x$

2. transitivity: $\forall\, x, y, z \in S$
   $x<y$ and $y<z \Rightarrow x<z$

An ordered set is a pair $(S,<)$ of a set and an order.

Ex. $\mathbb{Q}$ is an ordered set.
$\mathbb{Q}$ is also a field.

An ordered field is a field F such that

1. $\forall\, x, y, z \in F$, $y<z \Rightarrow y+x<z+x$
2. if $x>0$, $y>0$, then $xy>0$.

Ex. $\mathbb{Q}$ is an ordered field.

Let S be ordered, $E \subseteq S$. We say E is bounded above if $\exists\, \beta \in S$ such that $x \leq \beta \;\; \forall\, x \in E$. We call β an upper bound of E.

Let $E \subseteq S$ be bounded above. If $\exists\, \alpha \in S$ such that:
1. α is an upper bound of E
2. if β is an upper bound of E, then $\alpha \leq \beta$,

then α is the least upper bound or supremum of E
$\alpha = \sup(E)$.

Greatest lower bounds are defined similarly, called the infimum of E, $\inf(E)$.

Ex. $S_1 = \{p \in \mathbb{Q} \mid p = \frac{n}{3},\ n \in \mathbb{Z}\}$
$\rightarrow$ not bounded above or below

$S_2 = \{p \in \mathbb{Q} \mid p = \frac{3}{n},\ n \in \mathbb{Z},\ n \neq 0\}$

$-3 \leq \frac{3}{n} \leq 3$ $\rightarrow$ bounded above and below: $\sup(S_2) = 3$, $\inf(S_2) = -3$

$E_1 = \{\frac{1}{2}, 1, 2\} \in \mathbb{Q}$. $\inf(E_1) = \frac{1}{2}$, $\sup(E_1) = 2$

$E_2 = \{p \in \mathbb{Q} \mid p > 0\} = \mathbb{Q} > 0$. $\sup(E_2) \rightarrow$ DNE, $\inf(E_2) = 0 \in \mathbb{Q}$, $\notin E_2$

$E_3 = \{p \in \mathbb{Q} \mid p < 2\}$ $\sup(E_3) = 2$

$E_4 = \{p \in \mathbb{Q} \mid p \leq 2\}$ $\sup(E_4) = 2$

$E_5 = \{p \in \mathbb{Q} \mid p^2 < 2\}$ $\sup(E_5)$ DNE!

An ordered set S has the least-upper-bound property if every nonempty subset of S that has an upper bound also has a least upper bound in S.

Note: $\mathbb{Q}$ does not have the least-upper-bound property.

Theorem: $\mathbb{R}$ is an ordered field with the least-upper-bound property, and $\mathbb{Q}$ is a subfield of $\mathbb{R}$ ($\mathbb{Q} \in \mathbb{R}$, same $+$ and $\times$).

A cut (Dedekind cut) is a subset $\alpha \subseteq \mathbb{Q}$ such that

(a) α is nonempty and proper ($\alpha \neq \emptyset$, $\alpha \neq \mathbb{Q}$)
(b) if $p \in \alpha$ and $q \in \mathbb{Q}$, $q < p$, then $q \in \alpha$
(α is closed downward)
(c) if $p \in \alpha$, then $\exists\, r \in \alpha$ such that $p < r$
(no largest element)

Ex: $\{p \in \mathbb{Q} \mid p < 1\}$ is a cut
$\{p \in \mathbb{Q} \mid p \leq 2\}$ is not a cut

Def: $\mathbb{R} = \{\alpha \mid \alpha \text{ is a cut}\}$

Ex. $\{p \in \mathbb{Q} \mid p < 0 \text{ or } p^2 < 2\} \Rightarrow \sqrt{2}$


$\sqrt{2}$

Consequences of the least-upper-bound property:

Archimedian property: if $x, y \in \mathbb{R}$ and $x > 0$, then $\exists\, n \in \mathbb{N}$ such that $nx > y$.

Proof: Let $A = \{nx \mid n \in \mathbb{N}\}$. Suppose $nx \leq y$ $\forall\, n \in \mathbb{N}$ (y is an upper bound of A).
Since $\mathbb{R}$ has least-upper-bound property, $\exists\, \alpha = \sup(A) \in \mathbb{R}$. But $x > 0$, so $\alpha - x < \alpha$.

So $\exists\, m \in \mathbb{N}$ such that

$\alpha - x < mx$ ($\alpha - x$ is not an upper bound of A)

Then $\alpha < mx + x = (m+1)x \in A$.

Thus, α is not an upper bound of A, since $\alpha < (m+1)x$, so $\alpha \neq \sup(A)$.
Therefore it is not the case that $nx \leq y$ $\forall\, n \in \mathbb{N}$. So, $\exists\, n \in \mathbb{N}$ such that $nx > y$.