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4.16.14

$$f(x) = \begin{cases} x^{4/3}\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\mathbb{R}\setminus\{0\}$: $f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx}\left(x^{4/3}\sin\left(\frac{1}{x}\right)\right)$

$$= x^{4/3}\frac{d}{dx}\left(\sin\left(\tfrac{1}{x}\right)\right) + \frac{d}{dx}\left(x^{4/3}\right)\sin\left(\tfrac{1}{x}\right)$$
$$= x^{4/3}\cos\left(\tfrac{1}{x}\right)\left(-x^{-2}\right) + \tfrac{4}{3}x^{1/3}\sin\left(\tfrac{1}{x}\right)$$
$$= -x^{-2/3}\cos\left(\tfrac{1}{x}\right) + \tfrac{4}{3}x^{1/3}\sin\left(\tfrac{1}{x}\right)$$

At $x=0$: $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\frac{h^{4/3}\sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} h^{1/3}\sin\left(\tfrac{1}{h}\right) = 0$

$\lim_{h\to 0}$

$\left(-h^{1/3} \leq h^{1/3}\sin\left(\tfrac{1}{h}\right) \leq h^{1/3}\right)$

$f'(0)$ exists!

Thus, $f'(x)$ exists for all $x \in \mathbb{R}$,
i.e., f is differentiable.

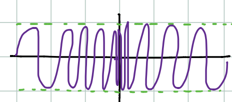
Is $f'(x)$ continuous?

$\rightarrow$ $f'(x)$ is the product of two continuous functions for $x \neq 0$, so $f'(x)$ is continuous $\forall\ x \neq 0$.

$f'(x)$ is continuous at $x=0$ iff $\lim_{x\to 0} f'(x) = f'(0) = 0$.

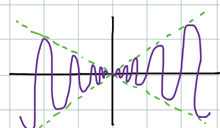
$\lim_{x\to 0} f'(x) \neq 0$, so $f'(x)$ is not continuous. $\left(x^{-2/3} \to -\infty \text{ as } x \to 0\right)$.

Note: $\begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$



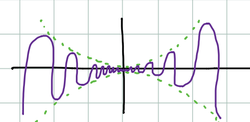
Not continuous at $x=0$.

$\begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$



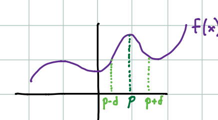
Continuous, not differentiable.

$\begin{cases} x^p\sin\left(\frac{1}{x}\right), & p>1, \quad x \neq 0 \\ 0, & x = 0 \end{cases}$



Continuous, differentiable,
$f'(x)$ not continuous.

Let $f: X \to \mathbb{R}$. Then f has a local maximum at $p \in X$ if $\exists\ \delta > 0$ such that $d(x,p) < \delta \Rightarrow f(x) \leq f(p)$.



Theorem: Let $f: [a,b] \to \mathbb{R}$. If f has a local max at $x \in (a,b)$,
Then $f'(x)$ exists and $f'(x) = 0$.

Proof: Choose δ from the definition of local max. at x such that
$a < x-\delta < x < x+\delta < b$.

If $x - \delta < t < x$, then $\frac{f(t)-f(x)}{t-x} \geq 0$.

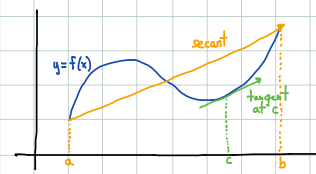
If $x < t < x+\delta$, then $\frac{f(t)-f(x)}{t-x} \leq 0$.

$f'(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x}$ exists. Thus $f'(x) \leq 0$ and $f'(x) \geq 0$. So $f'(x) = 0$. ∎

Corollary: $f: [a,b] \to \mathbb{R}$. f continuous on $[a,b]$, differentiable on (a,b). If $f(a) = f(b)$,
then $\exists\ x \in (a,b)$ such that $f'(x) = 0$. (Proof by IVT.)

Theorem: (Mean Value Theorem)

If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\ c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



Theorem: (Generalized MVT)

If f, g are cont. on $[a,b]$ and diff. on (a,b), $\exists\ c \in (a,b)$ such that

$$(f(b)-f(a))\,g'(c) = (g(b)-g(a))\,f'(c).$$

Note: If $g'(c) \neq 0$, $g(b)-g(a) \neq 0$, then $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{(f(b)-f(a))/(b-a)}{(g(b)-g(a))/(b-a)}$.

$g(x) = x \Rightarrow$ original MVT.

Proof: Consider $h(x) = (f(a)-f(b))\,g(x) - (g(b)-g(a))\,f(x)$.

$h(a) = h(b)$. So $\exists\ c \in (a,b)$ such that $h'(c) = 0$. ∎