Theorem: $\mathbb{Q}$ is dense in $\mathbb{R}$. Hannah Rose

If $x < y \in \mathbb{R}$, then $\exists\ q \in \mathbb{Q}$ such that

$$x < q < y.$$

Proof: If $x < y$, then $y - x > 0$. Then, by the Archemedian property, $\exists\ n_1 \in \mathbb{N}$ such that

$$n_1(y-x) > 1 \Rightarrow n_1 y > n_1 x + 1. \quad (1)$$

Also, by the Archemedian property, $\{nx\}$ is unbounded. Choose the first $n_2 \in \mathbb{N}$ such that

$$n_2 - 1 \le n_1 x < n_2. \quad (2)$$

Then $n_1 y > n_1 x + 1 \ge n_2$ from (1) and (2)

Thus $x < \frac{n_2}{n_1} < y$. $\therefore$, $\mathbb{Q}$ is dense in $\mathbb{R}$.

Ex. $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ denotes the extended real numbers.

$\Rightarrow \forall\ x \in \mathbb{R}$, $-\infty < x < +\infty$.

$$\left.\begin{array}{l} \bullet\ x + +\infty = +\infty \\ \bullet\ x + -\infty = -\infty \\ \bullet\ (+\infty) + (-\infty) = ? \\ \bullet\ \frac{1}{+\infty} = ? \qquad \frac{1}{-\infty} = ? \end{array}\right\} \text{NOT A FIELD}$$

The real Euclidean space of dimension n is

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}\}$$

$$x + y = (x_1 + y_1, \dots, x_n + y_n)$$

$$x \cdot y = (x_1 y_1, \dots, x_n y_n)$$

$(1, 1, \dots, 1, 0\}$ $\Rightarrow$ no mult. inverse $\Rightarrow$ NOT A FIELD (with this mult.)

Ex. $\mathbb{R}^2$.

$$(a,b) + (c,d) = (a+b, c+d)$$

$$(a,b) \cdot (c,d) = (ac - bd, ad + bc)$$

$$(0,0) + (a,b) = (a,b)$$

$$(1,0) \cdot (c,d) = (c - 0d, d + 0c) = (c,d)$$

Note: This is $\mathbb{C}$. It is a field.

$$(1,0) = 1$$

$$(0,1) = i$$

$$(a,b) = a + ib \quad \forall\ a,b \in \mathbb{R}$$

Ex. quaterions : $\mathbb{R}^4$

$$a + bi + cj + dk$$

where $i^2 = j^2 = k^2 = -1$

$ij = k$	$ji = -k$
$jk = i$	$kj = -i$
$ki = j$	$ik = -j$

Let $F: A \to B$ be a map of sets.

1. for $E \subseteq A$, the image of E under F is
$$F(E) = \{F(a) \mid a \in E\}$$
2. the range of F is $F(A)$
3. the domain is A
4. the codomain is B
5. F is onto or surjective if $F(A) = B$
6. the inverse image of $C \subseteq B$ is
$$F^{-1}(C) = \{a \in A \mid F(a) \in C\}$$
7. F is one-to-one or injective if $F(a) = F(a') \Leftrightarrow a = a'$
8. F is bijective if it is both injective and surjective
9. A and B are in one-to-one correspondence or have the same cardinality if there is a bijection $A \leftrightarrow B$.

*Note: $A \leftrightarrow B$ is an equivalence relation.

For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$.
Let A be a set.

a. A is finite if $A \leftrightarrow [n]$ for some n
b. A is infinite if it is not finite.
c. A is countable if $A \leftrightarrow \mathbb{N}$.
d. A is uncountable if it is not finite or countable.
e. A is at most countable if it is either finite or countable.

Let A be a set. A sequence in A is a function $F: \mathbb{N} \to A$.
$$F(n) = x_n.$$

Theorem: every subset of a countable set is at most countable.

Proof: Let A be countable, and $E \subseteq A$. If E is finite, then E is at most countable. Otherwise, E is infinite. Since A is countable, $\exists$ a bijection $A \leftrightarrow \mathbb{N}$. Let $\{a_n\}$ be this counting, i.e. $A = \{a_1, a_2, a_3 \ldots\}$.

Then all elements of E appear in the sequence $a_1, a_2, \ldots$. Let n_1 be the smallest $n \in \mathbb{N}$ such that $a_{n_1} \in E$. For $k > 1$, let n_k be the smallest integer greater than n_{k-1} such that $a_{n_k} \in E$.

This gives a map $\mathbb{N} \to E$. We have labeled all the elements of E using the elements of $\mathbb{N}$, so this is a bijection $\mathbb{N} \leftrightarrow E$. Thus, E is countable $\Rightarrow$ E is at most countable.