Hannah Rose
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Theorem: (1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(2) Let $A_1, A_2 \ldots \subseteq X$. Then $X \setminus \left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} (X \setminus A_i)$

and $X \setminus \left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} (X \setminus A_i)$.

Theorem: A countable union of countable sets is countable.

Proof: Let $\{A_1, A_2, \ldots\}$ be a list of countable sets, i.e. A_i is countable for each $i = 1, 2, \ldots$.
Our goal is to show that the union $\bigcup_{i=1}^{\infty} A_i$ is countable. Let

$$
\begin{aligned}
A_1 &= \{\overset{1}{a_{11}}, \overset{2}{a_{12}}, \overset{4}{a_{13}}, \ldots\} \\
A_2 &= \{\overset{3}{a_{21}}, \overset{5}{a_{22}}, a_{23}, \ldots\} \\
&\vdots \\
A_n &= \{a_{n1}, a_{n2}, a_{n3}, \ldots\} \\
&\vdots
\end{aligned}
$$

$\bigcup_{i=1}^{\infty} A_i = \{a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \ldots\}$. For $x \in \bigcup A_i$, $x \in A_n$ for some n, so $x = a_{nk}$ for some $k \in \mathbb{N}$.
Thus, the union is countable.

Theorem: Let A be countable. Then $A^n = \underbrace{A \times A \times \ldots \times A}_{n \text{ times}}$ is countable.

Proof: When $n = 1$, $A^1 = A$ is countable.
Suppose that A^k is countable. Note that A^{k+1} consist of

$$(x, y) = ((a_1, a_2, a_3, \ldots, a_k), a_{k+1})$$

where $x \in A^k$ and $y \in A$.

For a fixed $x_0 \in A^k$, $\{(x_0, y) \mid y \in A\}$ is countable (bijective with A).
Note that $A^{k+1} = \bigcup_{x \in A^k} \bigcup_{y \in A} \{(x, y)\} = \{(x, y) \mid x \in A^k, y \in A\}$. So A^{k+1} is a countable union of countable sets, and hence is countable by the previous theorem.

Corollary: $\mathbb{Q}$ is countable.

Proof: $\mathbb{Q} = \{(a, b) \mid a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\} \subseteq \mathbb{Z} \times \mathbb{Z}$.
So $\mathbb{Q}$ is at most countable. But $\mathbb{N} \subseteq \mathbb{Q}$, so $\mathbb{Q}$ is not finite, hence it must be countable.

An algebraic number is an element $x_0 \in \mathbb{R}$ such that

polynomial

$$\boxed{a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_n x_0^n} = 0$$

for some $a_0, a_n \in \mathbb{Z}$ (not all zero).

Theorem: The set of all algebraic numbers is countable.

→ how many polynomials of a given degree? (at most n roots)

Notation: Let $\{0,1\}^\infty = \{0,1\} \times \{0,1\} \times \dots = \{\text{infinite list of 0's and 1's}\} = \{f : \mathbb{N} \to \{0,1\}\}$.

Theorem: $\{0,1\}^\infty$ is uncountable.

Proof: Suppose $\{0,1\}$ is countable. $\{0,1\}^\infty = \{x_1, x_2, x_3, \dots\}$, where

$$x_1 = x_{11}\, x_{12}\, x_{13} \dots$$
$$x_2 = x_{21}\, x_{22}\, x_{23} \dots$$
$$x_3 = x_{31}\, x_{32}\, x_{33} \dots$$
$$\vdots$$

where $x_{ij} = \{0,1\}$.

Define $y = y_1 y_2 y_3 \dots$. Let $y_1 \neq x_{11}$, $y_2 \neq x_{22}$, and, in general, $y_n \neq x_{nn} = \{0,1\} \setminus \{x_{nn}\}$.
Then $y \in \{0,1\}^\infty$, $y \neq x_i$ for all $i \in \mathbb{N}$, since at least one element of y differs from every x_i.
Thus, y is not in $\{x_1, x_2, \dots\}$, so we have a contradiction and $\{0,1\}^\infty$ is not countable.

Remark: The same argument applies to decimal representatives of $\mathbb{R}$.

Other big sets:

The power set of the set A is the set of all subsets of A, denoted $P(A)$ or 2^A.

Theorem: If A is finite, $|P(A)| = 2^{|A|}$.

Theorem: For any set, $A \not\leftrightarrow 2^A$, A is not bijective with $P(A)$. (In fact, there is no surjection.)

Proof: Suppose $f: A \to P(A)$ is a surjection. Construct $B \subseteq A$, i.e. $B \in P(A)$ such that
$B \neq f(a)$ for any $a \in A$.

Let $B = \{a \in A \mid a \notin f(a)\}$. If B is in the image of f, then $\exists\, x \in A$
such that $f(x) = B$.

If $x \in B$, $x \notin f(x) \to x \notin B$, which is a contradiction.
If $x \notin B$, $x \in B$, which is also a contradiction.

Thus, $\nexists\, x \in A$ such that $f(x) = B$, so f is not surjective.