Fun fact: The Continuum Hypothesis

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2.5.14

We know $\mathbb{N}$ are countable. Let $\aleph_0$ be this cardinality. $\mathbb{R}$ is uncountable. Call its cardinality C.

Claim: there is no set with cardinality strictly between $\aleph_0$ and $C = 2^{\aleph_0}$.

$\rightarrow$ independent of axioms of set theory $\rightarrow$ <u>unprovable</u>.

$|\mathbb{Q}| = \aleph_0$

$|\mathbb{N}| = |\mathbb{N}^*|$

Metric Spaces

A <u>metric</u> on a set X is a relation

$$d : X \times X \longrightarrow \mathbb{R}$$

such that

(1) $\forall\ x, y \in X$

$\quad d(x,y) \geq 0$

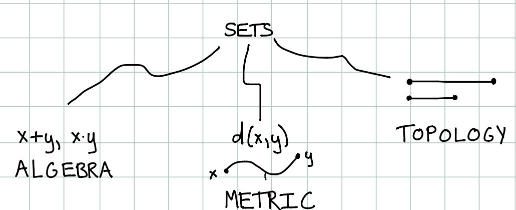
and $d(x,y) = 0$ iff $x = y$.

(2) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall\ x, y, z \in X$.

(3) $d(x,y) = d(y,x)$.

Remark: think distance.

<u>Big picture</u>:



A <u>metric space</u> is a pair (X,d) of a set and a metric.

Ex. (1) $\mathbb{R}$. $d(x,y) = |x-y|$

(2) $\mathbb{R}^2$. $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$ Euclidean distance

(3) $\mathbb{R}^n$. $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

(4) $\mathbb{R}^2$. $d(x,y) = \sum_{i=1}^{2} |x_i - y_i|$. Manhattan distance

(5) $X = \{\text{genetic sequences}\}$

$d(x,y) = $ # spaces in which they differ

(6) $X = $ any set

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$ discrete metric

Ex. Non-metrics

(7) $\mathbb{R}^n$. $d(x,y) = |x_3 - y_3| \longrightarrow$ fails $d(x,y) = 0$ iff $x=y$.

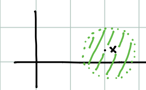
(8) $\mathbb{R}^2$. $d(x,y) = \sum x \ln\left(\frac{x}{y}\right) \longrightarrow$ fails symmetry

(9) X $d(x,y) = 0 \longrightarrow$ fails if $|X| > 1$.

Let X be a metric space. Let $r > 0$ $(r \in \mathbb{R})$. The <u>open ball</u> or <u>neighborhood</u> of $x \in X$ with radius r is

$$B_r^\circ(x) = B^\circ(x,r) = N_r(x)$$
$$= \{ y \in X \mid d(x,y) < r \}.$$

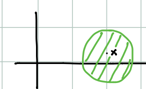
Ex. $\mathbb{R}^2$



The closed ball centered at x of radius r is

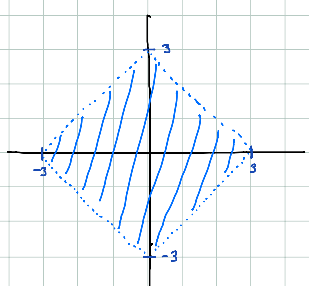
$$\overline{B}_r(x) = \overline{B}(x,r)$$
$$= \{ y \in X \mid d(x,y) \leq r \}$$

Ex. $\mathbb{R}^2$

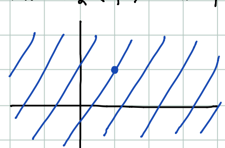


Ex. $\mathbb{R}^2$, Manhattan

$$N_3(0,0) = \{ (x,y) \in \mathbb{R}^2 \mid |x| + |y| < 3 \}$$

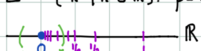


Ex. $N_2(1,1)$ $\mathbb{R}^2$, discrete



A point p in a metric space X is a <u>limit point</u> of a subset $E \subseteq X$ if for each neighborhood $N_r(p)$ of p there is a point $q \in X$ such that
$$q \neq p, \quad q \in E, \quad q \in N_r(p).$$

Ex. $X = \mathbb{R}$. $E = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$. $p = 0$



$N_r(0)$

Claim: 0 is a limit point of E.

Pf. $\exists\, n \in \mathbb{N}$ such that $\frac{1}{n} < r$. But the Archimedean property yields an $n \in \mathbb{N}$ such that $nr > 1$. Thus 0 is a limit point of E.

Let $p \in X$, $E \subseteq X$. Then p is an <u>interior point</u> of E
if $\exists\ r > 0$ such that $N_r(p) \subseteq E$.

The set of all interior points of E is called the <u>interior of E</u>, denoted $\underline{E^\circ}$.

Ex. $E = (-1, 2] \subseteq \mathbb{R}$.

$E^\circ = (-1, 2)$.

Suppose $p(x)$ is a property.

$\forall$ is a universal quantifier $\to$ $\forall x\ p(x)$

$\exists$ is an existential quantifier $\to$ $\exists x\ p(x)$

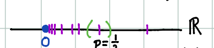
$\neg(\forall x\ p(x)) = \exists x\ (\neg p(x))$

A point $p \in X$ is an <u>isolated point</u> of $E \subseteq X$ if
$p \in E$ and p is <u>not</u> a limit point of E.

Note: p is not a limit point of E if $\neg(\forall N_r(p)\ \exists q \in N_r(p)$ such that $q \in E,\ q \neq p)$

$\exists N_r(p)\ \forall q \in N_r(p)$ such that $q \notin E$ or $q = p$.

Ex. $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$



↳ p is isolated

no isolated points