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2.10.14

Let $E \subseteq S$. The interior of E is $E^\circ = \{x \in E \mid x \text{ is an interior point of } E\}$.
So $x \in E^\circ \Leftrightarrow \exists\, r>0$ such that $N_r(x) \subseteq E$.

$E' = \{x \in E \mid x \text{ is a limit point of } E\}$. So $x \in E' \Leftrightarrow \forall\, r>0,\ \exists\, q \in N_r(x)$ such that $q \neq x,\ q \in E$.

A subset of a metric space, $E \subseteq X$, is open iff every point in E is an interior point of E.

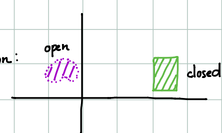
A set $E \subseteq X$ is closed iff $E' \subseteq E$.

Ex. $E = (1,2]$.

$E' = [1,2]$

$E^\circ = (1,2)$

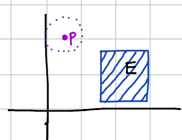
Intuition:



The closure of $E \subseteq X$ is $\bar{E} = E \cup E'$.

Theorem: $E \subseteq A$ is closed iff the complement of E, $A \setminus E = E^c$ is open.

Proof:



We must show that E^c is open, hence $(E^c)^\circ = E^c$.

Let $p \in E^c$. Since E contains all its limit points, p is not a limit point of E, $p \notin E'$. So $\exists\, N_r(p)$ such that $N_r(p) \cap E = \emptyset$. Thus, $N_r(p) \subseteq E^c$. So $p \in (E_c)^\circ$ for all $p \in E^c$. Thus E^c is open.

Now assume that E^c is open. Let $p \in E^c$. Then $p \in (E^c)^\circ$ because E^c is open by assumption. Thus $\exists\, r>0$ such that $N_r(p) \subseteq E^c$. Thus p is not a limit point of E, so all limit points of E are contained in E. Thus E is closed.

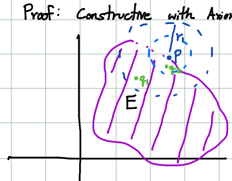
Note: $\emptyset$ is both open and closed.

Corallary: $E \subseteq X$ is open iff $E^c = X \setminus E$ is closed.

Note: X is both closed and open.

Theorem: If $p \in E'$, then every neighborhood of p contains infinitely many points of E.
Corallary: A finite set has no limit points.

Proof: Constructive with Axiom of Choice.



Let $r_1 > 0$. $\exists\, q_1 \in N_{r_1}(p) \cap E,\ q_1 \neq p$.
Let $r_2 = \frac{d(p,q_1)}{2}$. Then $\exists\, q_2 \in N_{r_2}(p) \cap E,\ q_2 \neq p$.
Given $r_{i-1} > 0$, $\exists\, q_i \in N_{r_{i-1}}(p) \cap E$ for $r_i = \frac{d(p, q_{i-1})}{2}$.

Alternate proof: Let $r_1 > 0$. Suppose $N_r(p) \cap E = \{q_1, \dots, q_n\}$ (a finite set).
Note that $p \neq q_i$ for any i.

Let $r' = \min\{d(p, q_i) \mid i = 1, \dots, n\}$. Then $\exists\, q' \in N_{r'}(p) \cap E$, but q' was not in the original set, so we have a contradiction.

Theorem: Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets. Then $U = \bigcup_{\alpha \in I} U_\alpha$ is an open set.

Proof: Let $p \in U$. Then $\exists\, \alpha$ such that $p \in U_\alpha$, which is open. So p is an interior point of U_α, i.e. $\exists$ $r > 0$ s.t. $N_r(p) \subseteq U_\alpha$. Thus $N_r(p) \subseteq U_\alpha \subseteq \bigcup_\alpha U_\alpha = U$. So $p \in U^\circ$. Therefore U is open.

Theorem: Let $U_1, U_2, \dots, U_n$ be open in X. Then $U_1 \cap U_2 \cap \dots \cap U_n$ is open in X.

Proof: If $\bigcap_{i=1}^n U_i = \emptyset$, then $\bigcap_{i=1}^n U_i$ is trivially open. Otherwise, let $p \in \bigcap_{i=1}^n U_i$. Then $p \in U_i$ for $i = 1, \dots, n$. But each U_i is open, hence $\exists\, r_i > 0$ such that $N_{r_i}(p) \subseteq U_i$.

Let $r = \min\{r_i \mid i = 1, \dots, n\}$. Then $N_r(p) \subseteq U_i$ for all i. Hence $N_r(p) \subseteq \bigcap_{i=1}^n U_i$. Thus $\bigcap_{i=1}^n U_i$ is open.

Ex. $X = \mathbb{R}$

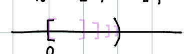
$E_n = (-\frac{1}{n}, \frac{1}{n})$

$\bigcap_{i=1}^\infty (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ (is not open)

Corollary: If $\{V_\alpha\}$ are closed, then $\bigcap_\alpha V_\alpha$ is closed.

$V_1, \dots, V_n$ are closed $\Rightarrow V_1 \cup V_2 \cup \dots \cup V_n$ is closed.

Proof: $V_n = [0, 1 - \frac{1}{n}]$, $n \in \mathbb{N}$.



$\bigcup_n V_n$

Let $\mathcal{T}$ be a collection of subsets of X. $\mathcal{T}$ is called a topology on X if

1) $\emptyset \in \mathcal{T}$
2) $X \in \mathcal{T}$
3) If $U_\alpha \in \mathcal{T}$ $\forall\, \alpha \in I$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$
4) If $U_1, U_2, \dots, U_n \in \mathcal{T}$ then $U_1 \cap U_2 \cap \dots \cap U_n \in \mathcal{T}$.

$(X, \mathcal{T})$ is a topological space iff $\mathcal{T}$ is a topology on X.

Elements of $\mathcal{T}$ are called open sets.