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2.12.14

Theorem: Let $E \subseteq X$. $\bar{E}$ is closed.

Proof: We will show that $X \setminus \bar{E}$ is open. Let $x \in X \setminus \bar{E}$.
Thus x is not a limit point of E. So $\exists\ N_r(x)$ such that $N_r(x) \subseteq X \setminus E$.

Suppose $\exists\ p \in N_r(x) \cap \bar{E}$. Then $p \in E'$; p is an interior point of $N_r(x)$. Thus $\exists\ N_{r'}(p) \subseteq N_r(x)$.
Since p is a limit point of E, $\exists\ q \in E$, $q \neq p$, $q \in N_{r'}(p)$.
So $q \in (N_{r'}(p) \cap E) \subseteq (N_r(x) \cap E) = \emptyset$. There are no elements of $\emptyset$, so there is no such q. Thus, $p \notin E'$,
so $N_r(x) \subseteq X \setminus \bar{E}$. Thus $x \in (X \setminus \bar{E})^\circ$, so $X \setminus \bar{E}$ is open.
Hence, $\bar{E}$ is closed.

Theorem: Let $E \subseteq X$. Then $E = \bar{E}$ iff E is closed.

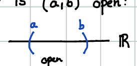
Proof: If $E = \bar{E}$, we have shown that E is closed. ✓

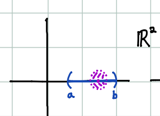
If E is closed, $E' \subseteq E$. Thus $E = E \cup E' = \bar{E}$. ✓

Theorem: Let $E \subseteq X$, $F \subseteq X$. If F is closed and $E \subseteq F$, then $\bar{E} \subseteq F$.

Proof: Let $p \in E'$. Consider $N_r(p)$. $\exists\ q \neq p$, $q \in E \cap N_r(p) \subseteq F \cap N_r(p)$.
Thus $p \in F' \subseteq F$. Therefore, $E' \subseteq F$, so $\bar{E} = E \cup E' \subseteq F$.

Q: is (a,b) open?



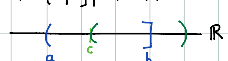


→ not open (no interior points)
For any $r > 0$, $N_r(p) \not\subseteq E$ for any $p \in E$.

Open is not an intrinsic property of E.

Let $Y \subseteq X$ be metric spaces. Then $E \subseteq Y$ is open relative to Y iff $E = Y \cap U$ for some open $U \subseteq X$.

Ex $Y = (a,b]$, $X = \mathbb{R}$.



(a,c) is open relative to Y because $U = (a,c)$ is an open set in X, and $U = U \cap Y$.

Let $U = (c, b+1)$. Then U is open in X and $(c, b+1) \cap (a,b] = (c,b]$.
So $(c,b]$ is open relative to Y.

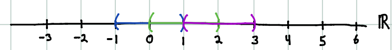
Q: Let X be a metric space. Is X open relative to X?
$\emptyset$ is closed, so $\emptyset^c = X$ is open. $\therefore$ $X = X \cap X$ is open relative to X.
(true for every topological space) Note: X is also closed.

Compact Sets

Let $E \subseteq X$, X is a metric space.

An open cover of E is a collection of sets $\{U_\alpha\}$, U_α is open in X $\forall\ \alpha$, and $E \subseteq \bigcup_\alpha U_\alpha$.

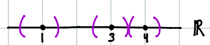
Ex. (a) $E = X = \mathbb{R}$. $U_n = (n-1, n+1)$, $n \in \mathbb{Z}$.



(b) $E = X = [0,1]$

$U_n = [1, \frac{1}{n})$ $\quad V_n = (1-\frac{1}{n}, 1]$ $\longrightarrow$ $\{U_n\} \cup \{V_n\}$ is an open cover

(c) $E = \{1, 3, 4\} \subseteq X = \mathbb{R}$



$U_n = (n-\frac{1}{2}, n+\frac{1}{2})$ for $n = 1, 3, 4$

A space $K \subseteq X$ is compact iff every open cover has a finite subcover.

A subcover of $\{U_\alpha\}$ is a subset of $\{U_\alpha\}$ that covers K.
It is finite if the number of sets is finite.

(a) not compact!
(b) yes, but we can't prove it yet.
(c) " " (almost obvious)