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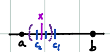
Goal: Heine - Borel Theorem

Given $X \subseteq \mathbb{R}^n$, X is compact iff X is closed and bounded.

Recall: Nested closed intervals have a non-empty intersection (same for k-cells).

k-cell : $[a_1, b_1] \times \ldots \times [a_k, b_k]$

Theorem: Every closed interval $[a,b] \subseteq \mathbb{R}$ is compact (and so are k-cells in $\mathbb{R}^k$).

Proof:  $\mathbb{R}$

Suppose $[a,b] \subseteq \mathbb{R}$ is not compact. Then $\exists$ an open cover $\{G_\alpha\}$ of $[a,b]$ with no finite subcover. Let $c_1 = (a+b)/2$. Then at least one of $[a, c_1]$ and $[c_1, b]$ has no finite subcover. WLOG, suppose it is $[a, c_1]$. Let $c_2 = (a+c_1)/2$.

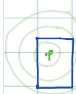
Let $I_1 = [a, c_1]$. Then at least one of $[a, c_2]$ or $[c_2, c_1]$ has no finite subcover. Call it I_2. Repeat this process to obtain $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$
Nested closed intervals have a nonempty intersection, so $\exists$ $x \in \bigcap_{n=1}^{\infty} I_n$.

Since $\{G_\alpha\}$ is an open cover of $[a,b]$, $\exists$ some α_0 such that $x \in G_{\alpha_0}$.
Thus, $\exists$ $r > 0$ such that $N_r(x) \subseteq G_{\alpha_0}$. For n large enough, $I_n \subseteq N_r(x) \subseteq G_{\alpha_0}$.
But I_n has no finite subcover, so a single G_{α_0} can't cover it.

Thus, $[a,b]$ is compact. ■

Heine-Borel Theorem: In $\mathbb{R}$, or $\mathbb{R}^n$, K is compact iff K is closed and bounded.

Proof: ($\Rightarrow$) Let $K \subseteq \mathbb{R}^n$ be compact. We have already proved that compact sets are closed. So we must show that K is bounded.

 K

Let $p \in K$. Then $K \subseteq \bigcup_{n=1}^{\infty} N_n(p)$. Indeed, $q \in K$. Then $\exists$ $n \in \mathbb{N}$ such that $d(p,q) < n \implies q \in N_n(p)$.

Since K is compact, $\exists$ a finite subcover $\{N_{n_1}(p), \ldots, N_{n_k}(p)\}$. Choose $r = \max\{n_1, \ldots, n_k\}$. Then $K \subseteq N_r(p)$. So K is bounded.

($\Leftarrow$) Suppose K is closed and bounded. That K is bounded means that $\exists$ $r > 0$ such that $K \subseteq [-r, r]$ (or k-cell in $\mathbb{R}^k$). Then K is a closed subset of a compact set ($[-r,r]$). Hence K is compact. ■

Ex. Let A be an infinite set.

$$d(p,q) = \begin{cases} 0 & p = q \\ 1 & p \neq q \end{cases}$$

$A \subseteq A$ is closed. A is also bounded

$A \subseteq N_2(p)$ for any $p \in A$.

But A is not compact.

Consider $\{N_{1/2}(p) \mid p \in A\}$. This has no finite subcovers, so A is not compact.

Remark: Under this metric, all subsets of A are open. (Each point is contained in its own neighborhood.)
(All subsets are also closed under this metric.)

Goal: Bolzanno – Weierstrass Theorem
Every infinite bounded subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Theorem: A metric space X is compact iff every infinite subset $E \subseteq X$ has a limit point in X.

Proof: $(\Rightarrow)$ We will prove the contrapositive. Suppose E has no limit points in X. Then each $q \in X$ has a neighborhood V_q which contains no other points of E. Then $\{V_q\}_{q\in X}$ is an open cover of X. There is no finite subcover because $|E| = \infty$. So X is not compact.

Hence, if X is compact, every infinite $E \subseteq X$ has a limit point in X.

$(\Leftarrow)$ $\mathbb{R}^k$. Assume every infinite subset $E \subseteq X$ has a limit point in X $(X \subseteq \mathbb{R}^k)$. We will show that X is closed and bounded, so X is compact by Heine-Borel

Suppose X is not closed. Then $\exists\ z \in X'$, $z \notin X$. Choose $E_n = \{x_n \mid |x_n - z| < \frac{1}{n}\} = N_{1/n}(z)$.

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