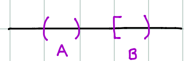
Hannah Rose
3/3/14

Let $A, B \subseteq X$. A and B are <u>separated</u> if $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.

$E \subseteq X$ is <u>connected</u> if it is not the union of two nonempty separated sets.
If E is not connected, it is called <u>disconnected</u>, and A and B are <u>separating sets</u> that form a <u>separation</u>.

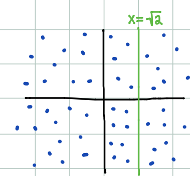
Note: $\emptyset$ is connected.

Ex.



$E = A \cup B \rightarrow$ disconnected

Ex. $E = \{(x,y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Q}\} = \mathbb{Q}^2 \subseteq \mathbb{R}^2$



$A = \{(x,y) \in E \mid x < \sqrt{2}\}$
$B = \{(x,y) \in E \mid x > \sqrt{2}\}$

$\overline{A} = \{(x,y) \in E \mid x \leq \sqrt{2}\}$
$\rightarrow \overline{A} \cap B = \emptyset = A \cap \overline{B}$

So E is disconnected.

Theorem: $[a,b] \subseteq \mathbb{R}$ is connected.

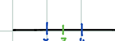
Proof: If $E = [a,b]$ is not connected, then $\exists$ a separation $[a,b] = A \cup B$ with $A \neq \emptyset$, $B \neq \emptyset$, $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.

Since A is bounded, $\sup(A)$ exists. Let $s = \sup(A)$. Then $s \in \overline{A}$.
Therefore $s \notin B$. We have $[a,b] = A \cup B$, so get $s \in A$.
Thus $s \notin \overline{B}$, so $\exists$ $\varepsilon > 0$ such that

$$N_\varepsilon(s) \subseteq [a,b] \setminus B = A.$$

But $s = \sup(A)$, so we have a contradiction. Thus, $[a,b]$ is connected.

Remark: (interval property)
$E \subseteq \mathbb{R}$ is connected iff $x, y \in E$, $\exists$ $z \in \mathbb{R}$, $x < z < y \Rightarrow z \in E$.



Theorem: Let $E \subseteq X$. The following are equivalent:

1. E is connected.
2. E is not the union of two disjoint, nonempty, open sets.
3. E is not the union of two disjoint, nonempty closed sets. $A \cup B = E \rightarrow A = E \setminus B$
4. The only subsets of E that are both open and closed relative to E are E and $\emptyset$.

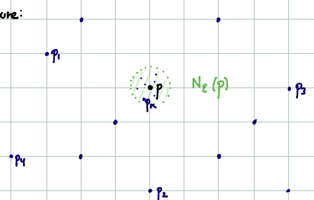
Sequences

A *sequence* in a metric space X is a function
$$F: \mathbb{N} \to X$$
$$n \to p_n$$
For each $p_n \in X$.

The sequence p_n *converges in X* if $\exists\, p \in X$ such that
$\forall\ \varepsilon > 0,\ \boxed{\exists\ N \in \mathbb{N}}$ such that $n \geq N$ implies $d(p_n, p) < \varepsilon$.
find to prove convergence

We say that $\{p_n\}$ *converges to p*, or that p is the limit of $\{p_n\}$.
Notation: $\lim_{n\to\infty} p_n = p$ or $p_n \to p$

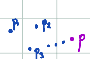
Picture:



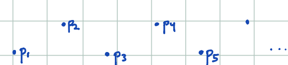
Note: for anyone using these notes (hi!), I recommend looking up the OEIS (online encyclopedia of integer sequences). It's pretty cool.

If $\{p_n\}$ does not converge to any point in X, we say that $\{p_n\}$ *diverges in X*.

Ex. (1)



(2)



(3) $\bullet\ p_n = p$

(4)

The range of $\{p_n\}$ is $\text{range}(\{p_n\}) = \{x \in X \mid x = p_n \text{ for some } n \in \mathbb{N}\}$
$\{p_n\}$ is bounded if $\text{range}(\{p_n\})$ is bounded in X.

T/F:

T A) $p_n \to p$, $p_n \to p'$, $p = p'$

F B) $\{p_n\}$ bounded $\Rightarrow$ $\{p_n\}$ converges

T C) $\{p_n\}$ converges $\Rightarrow$ $\{p_n\}$ bounded

F D) $p_n \to p \Rightarrow p \in (\text{range}(\{p_n\}))'$

T E) $p \in E'$, $E \subseteq X \Rightarrow \exists\ \{p_n\}$ in E, $p_n \to p$

T F) $p_n \to p \Leftrightarrow$ every neighborhood of p contains all but finitely many p_n