A. G. CLASS 10

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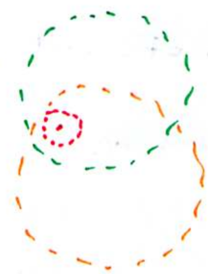
DEF (RECALL) IF X IS A SET, A BASIS FOR A TOPOLOGY ON X IS A COLLECTION $\mathcal{B}$ OF SUBSETS OF X SUCH THAT

(1) $\forall x \in X \; \exists B \in \mathcal{B}$ s.t. $x \in B$

(2) IF $x \in B_1 \cap B_2$, $B_1, B_2 \in \mathcal{B}$, THEN $\exists B_3 \in \mathcal{B}$ s.t.

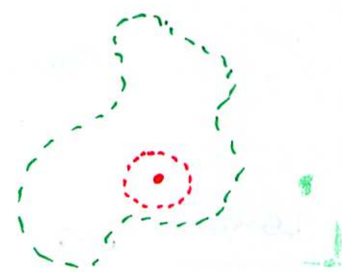
$$x \in B_3 \subseteq B_1 \cap B_2$$



DEF: IF $\mathcal{B}$ IS A BASIS FOR A TOPOLOGY ON X, THE TOPOLOGY $\tilde{\mathcal{T}}$ GENERATED BY $\mathcal{B}$ IS DEFINED BY:

LET $U \subseteq X$. $U \in \tilde{\mathcal{T}} \iff \forall x \in U \; \exists B \in \mathcal{B}$ s.t.

$$x \in B \subseteq U.$$



CLAIM: THE TOPOLOGY $\tilde{\mathcal{T}}$ GENERATED BY A BASIS IS A TOPOLOGY.

PF: (GROUPWORK)

EXAMPLE: $X = \mathbb{R}^2$. $\mathcal{B}_1 = \{\text{OPEN DISKS}\}$ $\mathcal{B}_2 = \{\text{OPEN RECTANGLE}\}$

$(x-a)^2 + (y-b)^2 \leq R^2$ $\qquad (a,b) \times (c,d)$

GROUPWORK: DESCRIBE/COMPARE TOPOLOGIES GENERATED BY $\mathcal{B}_1$, $\mathcal{B}_2$ & SHOW $\mathcal{B}_1$ & $\mathcal{B}_2$ ARE BASES FOR TOPOLOGIES ON $\mathbb{R}^2$.

LEMMA: LET $\mathcal{B}$ BE A BASIS FOR A TOPOLOGY $\mathcal{T}$ ON X. THEN $\mathcal{T}$ IS THE COLLECTION OF ALL UNIONS OF ELEMENTS OF $\mathcal{B}$.

PF: (GROUPWORK)

DEF: LET $\mathcal{T}, \mathcal{T}'$ BE TOPOLOGIES ON X. IF $\mathcal{T} \subset \mathcal{T}'$, WE SAY

~~LEMMA~~: $\mathcal{T}'$ IS FINER THAN $\mathcal{T}$. IF $\mathcal{T} \subsetneq \mathcal{T}'$ PROPER, THEN WE SAY $\mathcal{T}'$ IS STRICTLY FINER THAN $\mathcal{T}$. WE ALSO SAY $\mathcal{T}$ IS COARSER THAN $\mathcal{T}'$.

EX: $\mathcal{T} = \{\emptyset, X\}$, $\mathcal{T}' = \mathcal{P}(X)$. $\mathcal{T} \subseteq \mathcal{T}'$.

(THINK GRAVEL)

LEMMA: LET $\mathcal{B}, \mathcal{B}'$ BE BASES FOR $\mathcal{T}, \mathcal{T}'$ ON X.

TFAE

(1) $\mathcal{T}'$ IS FINER THAN $\mathcal{T}$

(2) $\forall x \in X$, IF $x \in B \in \mathcal{B}$, THEN $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$.

PF: (GROUPWORK)

EX: 

DEF: A MAP $f : (X, \tilde{\mathcal{T}}_X) \to (Y, \tilde{\mathcal{T}}_Y)$ BETWEEN TOPOLOGICAL SPACES IS CONTINUOUS IF $f^{-1}(U)$ IS OPEN IN X FOR ALL $U \in \mathcal{T}_Y$.

IE $V \in \tilde{\mathcal{T}}_Y \Rightarrow f^{-1}(V) \in \mathcal{T}_X$.

DEF: A BIJECTIVE MAP $f : X \to Y$ WHICH IS CONTINUOUS AND $f^{-1} : Y \to X$ IS ALSO CONTINUOUS IS CALLED A HOMEOMORPHISM.

DENOTED $X \cong Y$, X & Y CALLED HOMEOMORPHIC.

EX:

$S^1 \setminus$ pt.  $\cong$ $\mathbb{R}$ (ANALYTIC TOPOLOGY)