DAGAN KARP

A.G. CLASS 11

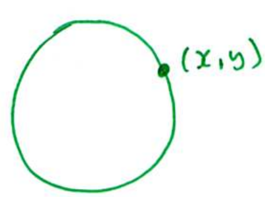
DEF: AN OPEN SUBSET OF AN AFFINE VARIETY IS CALLED A QUASI-AFFINE VARIETY.

EX: $\mathbb{A}^1_k \setminus \{0\}$  $\mathbb{A}^1_k \setminus Z(x)$

DEF: LET Y BE A QUASI-AFFINE VARIETY IN $\mathbb{A}^n$.

A FUNCTION $f: Y \to k$ IS REGULAR AT $p \in Y$ IF $\exists$ OPEN NBHD U OF p SUCH THAT $p \in U \subseteq Y$ AND $\exists$ POLYNOMIALS $g, h \in A = k[x_1, \ldots, x_n]$ SUCH THAT h IS NOWHERE ZERO ON U AND $f = g/h$ ON U. WE SAY f IS REGULAR ON Y IFF IT IS REGULAR AT EVERY POINT OF Y.

EX: $Y = Z(x^2+y^2-1)$



$f: Y \to k$

$(x,y) \mapsto \dfrac{1-y}{x}$

NEAR $(0,1)$, $f(x,y) = \dfrac{x}{1+y}$.

LEMMA: A REGULAR FUNCTION IS CONTINUOUS, WITH k IDENTIFIED AS $\mathbb{A}^1_k$ UNDER THE ZARISKI TOPOLOGY.

PF: LET $f: Y \to k$ BE REGULAR. WE SHOW $f^{-1}(Z)$ IS CLOSED FOR ANY CLOSED SUBSET $Z \subseteq \mathbb{A}^1_k$. BUT Z IS THEN A FINITE SET OF POINTS. SINCE THE FINITE UNION OF CLOSED SETS IS CLOSED, IT IS SUFFICIENT TO SHOW

$$f^{-1}(a) = \{ p \in Y \mid f(p) = a \}$$

IS CLOSED FOR ANY $a \in \mathbb{A}^1_k$.

WE USE A COOL TOPOLOGICAL FACT: THAT $f^{-1}(a)$ IS CLOSED CAN BE CHECKED LOCALLY, IE $f^{-1}(a)$ IS CLOSED IF AND ONLY IF $\exists$ A COLLECTION OF OPEN SETS $\{U_\alpha\}$ COVERING Y $(Y = \bigcup_\alpha U_\alpha)$ SUCH THAT $f^{-1}(a) \cap U_\alpha$ IS CLOSED IN U_α FOR ALL α.

LET $U \subseteq Y$ BE AN OPEN SET SUCH THAT $f = g/h$ ON U W/ $h \neq 0$ ON U. THEN

$$f^{-1}(a) \cap U = \{ p \in U \mid f(p) = g(p)/h(p) = a \}$$

BUT $\dfrac{g(p)}{h(p)} = a \Leftrightarrow \dfrac{g(p) - ah(p)}{h(p)} = 0 \Leftrightarrow g(p) - ah(p) = 0$

THUS $f^{-1}(a) \cap U = Z(g - ah) \cap U$ WHICH IS A CLOSED SUBSET OF U. BUT f IS REGULAR ON ALL OF Y, THUS Y IS COVERED BY SUCH SETS U. THUS $f^{-1}(a)$ IS CLOSED. THUS f IS CONTINUOUS. QED.

PROP: LET $X \subseteq \mathbb{A}^n$ BE ALGEBRAIC. TFAE. $(k = \bar{k})$

(1) X IS IRRED

(2) ANY TWO NONTRIVIAL OPEN SUBSETS OF X INTERSECT

$$U_1, U_2 \neq \emptyset, \quad U_1 \cap U_2 \neq \emptyset$$

(3) ANY NONEMPTY OPEN SET $U \subseteq X$ IS DENSE $(\bar{U} = X)$

(OR IF V ANY OPEN $U \cap V \neq \emptyset$)

PF: (GROUPWORK)

DEF: LET $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ BE ALGEBRAIC. A POLYNOMIAL MAP $f : X \to Y$ IS A MAP OF SETS SUCH THAT $\exists$ POLYNOMIALS

$$f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$$

SUCH THAT $f(p) = (f_1(p), \ldots, f_m(p)) \in \mathbb{A}^m_k$ FOR ALL $p \in X$.

DEF: A POLYNOMIAL MAP $f : X \to Y$ IS AN ISOMORPHISM

IF $\exists\ g : Y \to X$ SUCH THAT $f \circ g = \mathbb{1}_Y$, $g \circ f = \mathbb{1}_X$.

EX: $X = \mathbb{A}^1$, $Y = Z(y - x^2)$

THM: LET $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ BE ALGEBRAIC.

(1) LET $f : X \to Y$ BE A POLYNOMIAL MAP. THEN DEFINE

$$f^* : A(Y) \to A(X)$$

BY

$$f^*(g) = g \circ f$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f^* g"'] & Y \arrow[d, "g"] \\ & k \end{tikzcd} \quad g \in k[y_1, \dots, y_m]/I(Y)$$

THEN f^* IS A RING HOMOMORPHISM. (k-ALG HOM)

(2) ANY k-ALG HOM $\varphi : A(Y) \to A(X)$ IS OF THE FORM

$\varphi = f^*$ FOR SOME POLYNOMIAL MAP $f : X \to Y$.

(3) IF

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

ARE POLY MAPS, THEN

THEN $(g \circ f)^* = f^* \circ g^* : A(Z) \to A(X)$.

COR: BY (1) + (2)

$$\left\{ \begin{array}{c} \text{POLY MAPS} \\ X \to Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} k\text{-ALG HOM} \\ A(Y) \to A(X) \end{array} \right\}$$

$$f \qquad\qquad\qquad f^*$$

Pf: (1) (GROUPWORK)

COR: $f: X \to Y$ IS AN ISO. $\Longleftrightarrow$ $f^*: A(Y) \to A(X)$ IS AN ISO.

EX: $C = Z(y^2 - x^3) \subseteq \mathbb{A}^2$.

$$f: \mathbb{A}^1 \to C$$
$$t \mapsto (t^2, t^3)$$

C