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A. G. CLASS 12

RECALL: A POLYNOMIAL FUNCTION F ON AN ALGEBRAIC SET $X \subseteq \mathbb{A}^n_k$ IS SIMPLY A POLYNOMIAL $F: X \to k$, $F \in k[x_1,\ldots,x_n]$.

TWO POLYNOMIAL MAPS $F, G : X \to k$ ARE EQUAL ON X

$$f(p) = G(p) \quad \forall\, p \in X$$

IFF $F(p) - G(p) = 0 \;\; \forall\, p \in X \iff F - G \in I(X)$. SO THE COORDINATE RING OF X IS $A(X) = k[X] = A/I(X) = k[x_1,\ldots,x_n]/I(X)$.

A POLYNOMIAL MAP $F: X \to Y$ BETWEEN ALG. SUBSETS $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ IS A MAP OF SETS s.t. ~~$\forall p \in X$~~ $\exists\, F_1,\ldots,F_m \in A$ s.t.

$$F(p) = (F_1(p), \ldots, F_m(p)).$$

$F: X \to Y$ IS AN ISOMORPHISM IF $\exists$ POLY MAP $G: Y \to X$ s.t. $F \circ G = 1_Y$ $\;G \circ F = 1_X$. EACH POLY MAP $f: X \to Y$ INDUCES $f^*: k[Y] \to k[X]$, ANY k-ALG HOM $\varphi: A(Y) \to A(X)$ IS $\varphi = f^*$ FOR SOME POLY MAP $f: Y \to X$.

NOTE: (IDEA OF PF)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f^*g"'] & Y \arrow[d, "g"] \\ & k \end{tikzcd}$$

$f^*g = g \circ f$

$$A(Y) = k[y_1,\ldots,y_m]/I(Y)$$
$$\downarrow f^*$$
$$A(X) = k[x_1,\ldots,x_n]/I(X)$$

COMPOSITIONS OF POLYNOMIALS ARE POLYNOMIALS, SO $g \circ f \in A(X)$.

LET $g_1, g_2 \in A(Y)$, $p \in X$.

$$f^*(g_1+g_2)(p) = (g_1+g_2)\circ f(p) = g_1(f(p)) + g_2(f(p))$$

$$= f^*g_1(p) + f^*g_2(p)$$ POINTWISE RING STRUCTURE

$$\Rightarrow \quad f^*(g_1+g_2) = f^*g_1 + f^*g_2 \in A(X)$$

SIMILARLY FOR $f^*(g_1 g_2) = f^*g_1 \, f^*g_2$.

NOW, LET $\varphi : A(Y) \to A(X)$ BE A k-ALG HOM. IE

$\forall\, c \in k,\ g_1, g_2 \in A(Y)$

$$\varphi(cg_1) = c\,\varphi(g_1)$$

$$\varphi(g_1+g_2) = \varphi(g_1) + \varphi(g_2)$$

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2).$$

$\Rightarrow \varphi(g(y)) = g(\varphi(y))$

$g = \sum a_I y^I$

$\varphi(g) = \ldots$

WE MUST FIND $f : X \to Y$, POLYNOMIAL MAP SUCH THAT

$\varphi = f^*$.

COORDINATE FUNCTION $y_i(a_1, .., a_m) = a_i$

$$k[y_1,..,y_m] \xrightarrow{} A(Y) = k[y_1,..,y_m]/I(Y) \xrightarrow{\varphi} A(X) = k[x_1,..,x_n]/I(X)$$

LET $f_j = \varphi \circ y_j \in A(X)$ AND DEFINE

$$f : X \to \mathbb{A}^m$$
$$p \mapsto (f_1(p), .., f_m(p))$$

THEN f IS A POLY. MAP SINCE $f_j \in A(X)\ \forall j$.

CLAIM: $f(X) \subseteq Y$.

$Y = Z(I(Y))$. WE SHOW $f(X) \subseteq Z(I(Y))$. LET $p \in X$, $g \in I(Y)$.

WE SHOW $g(f(p)) = 0$.

$$g \in I(Y) \Rightarrow g = 0 \in A(Y) \Rightarrow \varphi(g) = 0 \in A(X)$$

k-ALG HOM.

BUT $\varphi(g(y_1, ..., y_m)) = g(\varphi(y_1), .., \varphi(y_m)) = g(f_1, .., f_m)$

THUS $g(f_1(p), .., f_m(p)) = 0 \ \forall p \in X$. THUS $g \in I(Y)$. THUS

$f(X) \subseteq Z(I(Y))$. NOW CHECK $f^*(y_j) = \varphi(y_j)$. QED

<u>COR.</u> $f: X \to Y$ POLY MAP IS AN ISO $\Leftrightarrow f^*: A(Y) \to A(X)$ IS AN ISO.

EX $Y = Z(y^2 - x^3)$ $X = \mathbb{A}^1$ $t \mapsto (t^2, t^3)$

$Y = Z(y - x^2)$ $X = \mathbb{A}^1$

$\varphi: k[Y] \to k[X]$ MAKE EXPLICIT, FIND INDUCED $\varphi = f^*$.

$\varphi: k[x,y]/(y - x^2) \to k[t]$

DEF: LET X BE AN AFFINE VARIETY. A RATIONAL FUNCTION ON X IS AN ELEMENT OF THE QUOTIENT FIELD / FIELD OF FRACTIONS / FUNCTION FIELD

IE $h \not\equiv 0$, h IS NOT THE ZERO POLY ON X.

$$k(X) = \{ g/h \mid g, h \in k[X],\ h \neq 0 \}/\sim$$

$$g/h \sim g'/h' \iff gh' = g'h \in k[X].$$

NOTE: THAT h IS NOT THE ZERO POLYNOMIALS ALLOWS FOR h TO HAVE SOME ZEROS IN X. THUS $f \in k(X)$ DOES NOT GIVE A MAP $f: X \to k$, AS IT IS NOT DEFINED ON ALL OF X. WE WRITE $f: X \dashrightarrow k$.

DEF: WE SAY f IS REGULAR AT $p \in X$ IF $\exists\, g, h$ S.T. $f(p) = g(p)/h(p)$ AND $h(p) \neq 0$. THE DOMAIN OF f IS

$$\mathrm{Dom}(f) = \{ p \in X \mid f \text{ IS REGULAR AT } p \}.$$

NOTATION: $\mathcal{O}_X = \{ f \in k(X) \mid f \text{ IS REGULAR ON ALL OF } X \}$

$\mathcal{O}_{X,p} = \{ f \in k(X) \mid f \text{ IS REGULAR AT } p \}$

REMARK: $\mathcal{O}_{X,p} \subseteq k(X)$ IS CALLED THE LOCAL RING OF X AT p.

$$\mathcal{O}_{X,p} = k[X][\{ h^{-1} \mid h(p) \neq 0 \}]$$

UNIQUE MAX'L IDEAL m_p.

THM: (1) $\mathrm{DOM}(f) \subseteq X$ IS OPEN AND DENSE IN ZARISKI

LET $k = \bar{k}$.

(2) $\mathrm{DOM}(f) = X \Leftrightarrow f \in k[X]$ (REGULAR RATIONAL FUNCTIONS ARE POLYNOMIAL FUNCTIONS)

(3) FOR $h \in k[X]$, LET

$$X_h = X \setminus Z(h) = \{ p \in X \mid h(p) \neq 0 \}$$

$$X_h \subseteq \mathrm{DOM}(f) \Leftrightarrow f \in k[X][h^{-1}].$$

DEF: LET $f \in k(X)$. THE IDEAL OF DENOMINATORS OF f IS

$$D_f = \{ h \in k[X] \mid hf \in k[X] \} \subseteq k[X]$$

$$= \{ h \in k[X] \mid \exists\, g \in k[X] \text{ s.t. } f = g/h \} \cup \{0\}$$

PF:

(1) D_f IS AN IDEAL, THUS

$$X \setminus \mathrm{Dom}(f) = \{ p \in X \mid h(p) = 0 \ \forall\, h \in D_f \} = Z(D_f)$$

THUS $\mathrm{DOM}(f)$ IS OPEN. IT IS NONEMPTY b/c $1 \in k[X]$.

THEREFORE IT IS DENSE.

(2) $\mathrm{DOM}(f) = X \Leftrightarrow Z(D_f) = \emptyset \overset{\text{NSS.}}{\Leftrightarrow} 1 \in D_f \Leftrightarrow f \in k[X]$.

(3) $X_h \subseteq \mathrm{DOM}(f) \Leftrightarrow h(p) \neq 0 \Rightarrow f$ REG AT p

$\Leftrightarrow f$ NOT REG AT $p \Rightarrow h(p) = 0$

BUT f NOT REG AT p $\Leftrightarrow$ $p \in Z(D_f)$

SO $p \in Z(D_f) \Rightarrow h(p) = 0$

IE $h \in I(Z(D_f)) \overset{\text{NSS}}{\Longleftrightarrow} h^n \in D_f$ FOR SOME $n \in \mathbb{N}$

$\Rightarrow f = g/h^n \in k[x][h^{-1}]$. QED