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A.G. CLASS 13

RECALL (FROM READING): LET V BE AN AFFINE VARIETY. A **RATIONAL MAP** $f: V \dashrightarrow \mathbb{A}^n_k$ IS A PARTIALLY DEFINED MAP GIVEN BY

$$f(p) = (f_1(p), \ldots, f_n(p)) \quad \forall\, p \in \bigcap_{i=1}^{n} \mathrm{DOM}(f_i)$$

WHERE EACH f_i IS A RATIONAL FUNCTION $f_i : V \dashrightarrow k$.

THEN $\underline{\mathrm{DOM}(f) = \bigcap \mathrm{DOM}(f_i)}$, AND f IS **REGULAR** AT $p \Leftrightarrow p \in \mathrm{DOM}(f)$.

IF $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ ARE VARIETIES, A RAT'L MAP

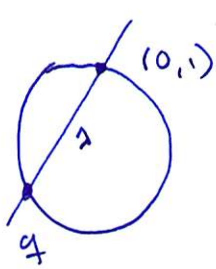
$$f: V \dashrightarrow W$$

IS A RAT'L MAP $f: V \dashrightarrow \mathbb{A}^m$ SUCH THAT $f(\mathrm{DOM}(f)) \subseteq W$.

EX: $\mathbb{R}^1 \to V = Z(y^3 - x^2)$ $\qquad V \dashrightarrow \mathbb{R}^1$

$$(t^2, t^3) \dashrightarrow t$$

$$(x, y) \dashrightarrow y/x$$



$$S^1 \dashrightarrow \mathbb{R}$$

$$\mathbb{R} \dashrightarrow S^1$$

$$L_\lambda \mapsto \frac{(2\lambda, \lambda^2 - 1)}{\lambda^2 + 1}$$

NOTE: $V \overset{f}{\dashrightarrow} W \overset{g}{\dashrightarrow} U$ $g \circ f$ MAY NOT BE DEFINED

$\mathbb{A}^1 \dashrightarrow \mathbb{A}^2 \dashrightarrow \mathbb{A}^1$

$x \qquad (x,0)$

$\qquad (x,y) \longmapsto x/y$

$\underline{\mathrm{DOM}(g \circ f) = \mathrm{DOM}(f) \cap f^{-1}(\mathrm{DOM}(g))}$

$\mathrm{DOM}(g) = \{(x,y) \mid y \neq 0\} \qquad f^{-1}(\mathrm{DOM}(g)) = \emptyset.$

REMARK: GIVEN $f: V \dashrightarrow W$, WE GET A k-ALG. HOM

$$f^*: k[W] \to k(V)$$

WITH $f^* g = g \circ f$. BUT IF $h \in \ker(f^*)$, THEN

$$f^*(g/h)$$

HAS NO MEANING / ISN'T DEFINED.

DEF: $f: V \dashrightarrow W$ IS DOMINANT IF $f(\mathrm{DOM}(f)) \subseteq W$ IS DENSE.

NOTE: f DOMINANT $\iff f^*: k[W] \to k(V)$ IS INJECTIVE.

Pf: $g \in \ker f^* \iff f(\mathrm{DOM}(f)) \subseteq Z(g)$

NOT INJECTIVE

PROPER ALGEBRAIC SUBSET OF W

f DOMINANT $\Rightarrow$ $g \circ f$ IS DEFINED.

THM: (I) DOMINANT RAT'L MAP $f : V \dashrightarrow W$ DEFINES FIELD HOM

$$f^* : k(W) \to k(V)$$

(II) k-HOM $\varphi : k(W) \to k(V)$ COMES FROM UNIQUE

DOMINANT RAT'L MAP $f : V \dashrightarrow W$

(III) f, g DOMINANT $\Rightarrow (g \circ f)^* = f^* \circ g^*$

<u>DEF:</u> LET V, W BE AFFINE VARIETIES, $U \subseteq V$ QUASI-AFFINE (OPEN SUBSET)

A MORPHISM $f : U \to W$ IS A RAT'L MAP $f : V \dashrightarrow W$ SUCH THAT $U \subseteq \mathrm{DOM}(f)$, IE f REGULAR ON U.

$$\{\text{MORPHISMS } V \to W\} = \{\text{POLY MAPS } V \to W\}$$

DEF: LET V BE AFFINE, $f \in k[V]$. THE STANDARD OPEN SUBSET ASSOCIATED TO f IS

$$V_f = V \setminus Z(f) = \{p \in V \mid f(p) \neq 0\}.$$

PROP: V_f IS ISO. TO AN AFFINE VARIETY,

$$k[V_f] \cong k[V][f^{-1}].$$

PF!

LET $J = I(V) \subseteq k[x_1,\dots,x_n]$. CHOOSE $F \in k[x_1,\dots,x_n]$ s.t.

$f = F$ MOD J. DEFINE

$$I = (J, yF-1) \subseteq k[x_1,\dots,x_n,y]$$

AND LET $W = Z(I) \subseteq \mathbb{A}^{n+1}$.

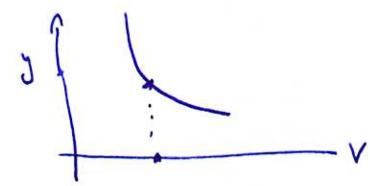
DEFINE

$$g : W \to V_f$$

$$(x_1,\dots,x_n,y) \mapsto (x_1,\dots,x_n)$$

$$(x_1,\dots,x_n,F^{-1}(x)) \leftarrow\!\!\shortmid (x_1,\dots,x_n)$$

$W = \{yF=1\}$



CLAIM: STANDARD OPEN SETS FORM A BASIS FOR THE ZARISKI TOPOLOGY.

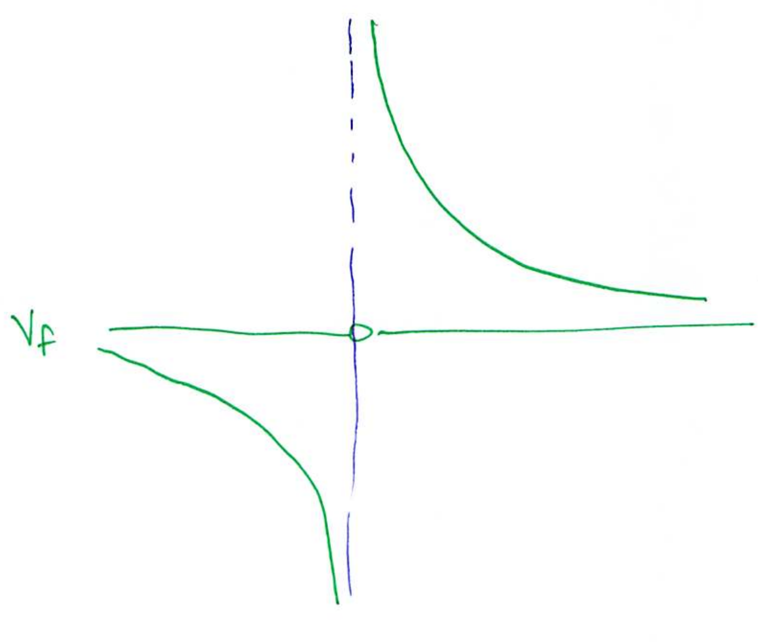
EX! $V = \mathbb{A}^1$ $f(x) = x$ $V_f = \mathbb{A}^1 \setminus \{0\}$

$= 0 \in$

$I(V) = k[x] = J$

$I = (J, yx-1)$

$W = Z(I)$



$(a,b) \in W \Leftrightarrow a \neq 0,\ b = 1/a$

$f(a)=0$ $f \in k[x]$