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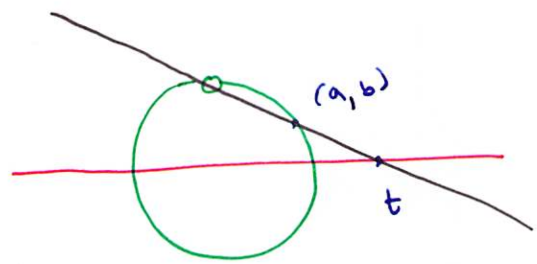
DEF: LET X AND Y BE AFFINE OR PROJECTIVE VARIETIES.

A RATIONAL MAP $f: X \dashrightarrow Y$ IS BIRATIONAL (OR A BIRATIONAL EQUIVALENCE) IF $\exists$ RATIONAL MAP $g: Y \dashrightarrow X$ SUCH THAT $f \circ g = \mathbb{1}_Y$ $g \circ f = \mathbb{1}_X$.

EX: STEREOGRAPHIC PROJECTION



$X = Z(x^2+y^2-1)$ $\quad Y = \mathbb{A}^1$

$f: X \dashrightarrow Y \quad f(x,y) = \frac{x}{1-y}$

$g: Y \dashrightarrow X \quad g(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \right)$

(~~CHECK~~ GROUP $f \circ g$, $g \circ f$) THUS S^1 AND $\mathbb{A}^1$ ARE BIRATIONALLY EQ.

PROP: LET $f: X \dashrightarrow Y$ BE A RAT'L MAP. TFAE

① f IS A BIRAT'L EQ.

② f IS DOMINANT AND $f^*: k(Y) \to k(X)$ IS AN ISO.

③ $\exists$ OPEN SETS $U \subseteq X$, $V \subseteq Y$ SUCH THAT $f|_U : U \to V$ IS AN ISOMORPHISM.

PF: (1) $\Leftrightarrow$ (2) AS BEFORE. WE SHOW (1) $\Leftrightarrow$ (3).

NOTE (3) $\Rightarrow$ (1) IMMEDIATELY, SINCE INVERSE MORPHISMS ON OPEN SETS COMPRISE A BIRATIONAL EQ.

(1) $\Rightarrow$ (3). WE MUST FIND U, V. (NOT QUITE $\mathrm{DOM}(f)$, $\mathrm{DOM}(g)$).

WE HAVE $f: X \dashrightarrow Y$, $g: Y \dashrightarrow X$, $f \circ g = \mathbb{1}_Y$, $g \circ f = \mathbb{1}_X$

LET $\varphi = f|_{\mathrm{DOM}(f)}: \mathrm{DOM}(f) \to Y$, $\psi = g|_{\mathrm{DOM}(g)}: \mathrm{DOM}(g) \to X$.

THEN φ, ψ ARE MORPHISMS. CONSIDER

$$\begin{tikzcd}
\psi^{-1}(\mathrm{DOM}(f)) \arrow[r, "\psi"] \arrow[d, phantom, "\cap|"] & \mathrm{DOM}(f) \arrow[r, "\varphi"] & Y \\
Y \arrow[urr] & &
\end{tikzcd}$$

$$\mathbb{1}_Y|_{\psi^{-1}(\mathrm{DOM}(f))} = \varphi \circ \psi \quad \text{b/c } \mathbb{1}_Y = f \circ g.$$

THUS

$$\varphi \circ \psi(p) = p \quad \forall p \in \psi^{-1}(\mathrm{DOM}(f)).$$

NOW LET $U = \varphi^{-1}\psi^{-1}(\mathrm{DOM}(f))$, $V = \psi^{-1}\varphi^{-1}(\mathrm{DOM}(g))$.

THEN $\varphi: U \to \psi^{-1}(\mathrm{DOM}(f))$ IS A MORPHISM. WHAT IS ITS RANGE?

LET $p \in U$. THEN $\varphi(p) \in \psi^{-1}\mathrm{DOM}(f)$. SO $\psi(\varphi(p)) \in \mathrm{DOM}(f)$

BUT $\psi(\varphi(p)) = p$. SO $p \in \psi^{-1}\varphi^{-1}(\mathrm{DOM}(g)) = V$.

THUS $\varphi : U \to V$ IS A MORPHISM. SIMILARLY FOR $\psi : V \to U$. QED.

COR: LET X BE A VARIETY. TFAE.

(a) $k(X) \cong k(t_1, \dots, t_n)$ FOR SOME $n \in \mathbb{N}$, IE $k(X)/k$ IS A PURELY TRANSCENDENTAL EXTENSION

(b) $\exists$ DENSE OPEN $U \subseteq X$ WHICH IS ISOMORPHIC TO A DENSE OPEN SUBSET $V \subseteq \mathbb{A}^n$.

DEF: SUCH A VARIETY IS CALLED RATIONAL.

NOTE: A RATIONAL VARIETY CAN BE PARAMETRIZED BY n INDEPENDENT VARIABLES.

EX: CUBICS IN $\mathbb{R}^2$

(i) $X_1 = Z(y^2 - x^3 + x^2) \subseteq \mathbb{R}^2$  $t \mapsto (t^2 - 1, t^3 - t)$

(ii) $X_2 = Z(y^2 - x^3)$  $t \mapsto (t^2, t^3)$

(iii) $X_3 = Z(y^2 - x(x-1)(x-\lambda))$, $\lambda \neq 0, 1$  NONE.

LET'S STUDY CUBICS: MODULI

NOTATION: LET $S_d = \{\text{FORMS OF DEGREE } d \text{ IN } x, y, z\}$

NOTE: S_d IS A VECTOR SPACE OVER k. WHAT IS ITS DIMENSION? (GROUP)

$F \in S_d \iff F = \sum_{i+j+k=d} a_{ijk} x^i y^j z^k$. $\quad \dim = \binom{d+2}{2}$

NOTATION: $S_d(p_1, .., p_n) = \{F \in S_d \mid F(p_i) = 0 \ \forall i = 1, .., n\} \subseteq S_d$. $\quad p_i \in \mathbb{P}^2$

Q: WHAT IS THE DIMENSION OF $S_d(p_1, .., p_n)$? $\geq \binom{d+2}{2} - n$ GW

OBSERVATION: LET k BE INFINITE, $F \in S_d$.

(1) LET $L \subseteq \mathbb{P}^2_k$ BE A LINE. IF $F = 0$ ON L, THEN $L = Z(H)$.

$F = H \cdot F'$, $F' \in S_{d-1}$.

(2) LET $C \subseteq \mathbb{P}^2_k$ BE A NONDEG CONIC, $C = Z(Q)$.

$F = 0$ ON $C \Rightarrow F = QF''$, $F'' \in S_{d-2}$.

(WHY? NSS) GW

COR: LET $p_1, .., p_n \in \mathbb{P}^2_k$ BE GIVEN, FIX d ALSO.

(1) IF $p_1, .., p_m \in L$, $p_{m+1}, .., p_n \notin L$, $m > d$, THEN

$S_d(p_1, .., p_n) = H \cdot S_{d-1}(p_{m+1}, .., p_n)$

(2) $p_1, \dots, p_m \in C$, $p_{m+1}, \dots, p_n \notin C$, $m > 2d$, THEN

$$S_d(p_1, \dots, p_n) = Q \, S_{d-2}(p_{m+1}, \dots, p_n)$$

BEZOUT: $L = Z(H) \subseteq \mathbb{P}^2$ LINE, $C = Z(Q) \subseteq \mathbb{P}^2$ NONDEG CONIC

$D = Z(G)$, $G \in S_d$. $L \not\subseteq D$, $C \not\subseteq D$.

$\# \, L \cap D \leq d$ $\quad\quad$ $\# \, C \cap D \leq 2d$

W/ EQUALITY IF COUNTED W/ MULTIPLICITY AND $k = \bar{k}$.