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A.G. CLASS 17

BEZOUT'S THM.

LET X BE A CURVE OF DEGREE d IN $\mathbb{P}^2_k$, AND Y BE A CURVE OF DEGREE e. IF $Y \neq \cancel{X}$, THEN

$Y \not\subseteq X$

$X \not\subseteq Y$

LET

$$\# X \cap Y$$

BE THE NUMBER OF POINTS IN $X \cap Y$ COUNTED WITH MULTIPLICITY.

THEN

$$\# X \cap Y \leq de$$

WITH EQUALITY IF k IS ALGEBRAICALLY CLOSED.

PF:

HOW TO DETERMINE IF TWO POLYNOMIALS IN ONE VARIABLE HAVE A COMMON ZERO OR COMMON FACTOR? THE RESULTANT.

LET $X = Z(f)$ $\quad f(x,y,z) = a_0 z^d + a_1 z^{d-1} + \dots + a_{d-1} z + a_d$

$Y = Z(g)$ $\quad g(x,y,z) = b_0 z^e + b_1 z^{e-1} + \dots + b_e$

WHERE a_i, b_i ARE HOMOG. POLY'S IN (x,y) OF DEGREE i,

SO F IS A d-FORM AND g IS AN e-FORM IN (x,y,z).

THE RESULTANT OF f, g IS THE DETERMINANT OF THE SYLVESTER MATRIX

$$\begin{array}{cc} d+e & \\ S(f,g) = \begin{pmatrix} a_0 & a_1 & \cdots & a_d & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_d & \cdots & 0 \\ \vdots & & & & & & \\ 0 & \cdots & & 0 & a_0 & \cdots & a_d \\ b_0 & b_1 & \cdots & & b_e & 0 & \cdots \\ \vdots & & & & & & \\ 0 & \cdots & & b_0 & \cdots & & b_e \end{pmatrix} & d+e \end{array}$$

THEN $|S(f,g)| = 0 \iff f, g$ HAVE A COMMON SOLUTION.

WHY? LET $P_n = \{$ POLYNOMIALS OF DEGREE $< n\}$

THIS IS A VECTOR SPACE OF DIM n W/ BASIS

$(1, z, z^2, .., z^{n-1})$.

CONSIDER THE MAP

$$P_d \times P_e \xrightarrow{\varphi} P_{d+e}$$

GIVEN BY $(p, q) \mapsto pg + qf$.

THEN φ IS LINEAR AND $[\varphi] = S(f,g)$. NOTE THAT φ IS NOT INJECTIVE $\iff \varphi$ HAS NONTRIVIAL KERNEL

$\iff [\varphi]$ IS NOT INVERTIBLE $\iff |S| = 0$.

IN FACT, IF f, g ARE MONIC ($a_0 = b_0 = 1$) THEN

$$|S| = \prod_{\substack{(z_1, z_2) \\ f(z_1) = g(z_2) = 0}} (z_1 - z_2)$$

EX: $f(z) = z - a$ $g(z) = z - b$

$$|S| = \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} = a - b$$

$|S| = 0 \iff a = b \iff f + g$ HAVE COMMON ZERO.

BACK TO BEZOUT: $|S|$ IS A HOM. POLY OF DEGREE de IN x, y.

THM NOW FOLLOWS FROM FTOA. QED.

* RETURN TO LAST CLASS: §d

PROP: LET $|k| = +\infty$, $p_1, .., p_8 \in \mathbb{P}^2_k$ s.t. $\{p_i\}$ DISTINCT, NO 4 COLLINEAR, NO 7 CONCONIC (ON A CONIC NON-DEG).

$$\dim S_3(p_1, .., p_8) = 2. \qquad (\text{KNOW } \geq 2)$$

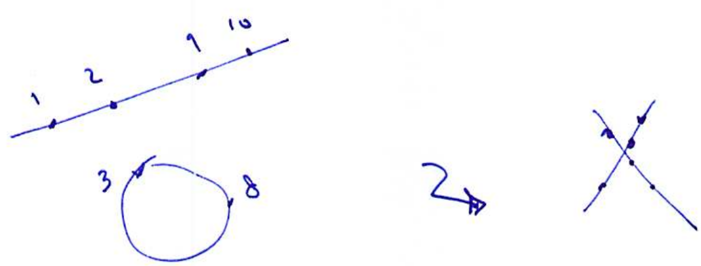
PF: NON DEG CASE: NO 3 COLLINEAR, NO 6 CONCONIC.

LET $p_9 \neq p_{10}$ BE DISTINCT PTS ON THE LINE $L = \overline{p_1 p_2}$

THEN SUPPOSE $\dim S_3(p_1,..,p_8) \geq 3$.

$$\dim S_3(p_1, .., p_{10}) \geq \dim S_3(p_1, .., p_8) - 2 \geq 1$$

$$\Rightarrow \quad \exists\, 0 \neq F \in S_3(p_1, .., p_{10}) \Rightarrow L \subseteq Z(F) \Rightarrow F = H \cdot Q$$



SIMILAR ARGS IN DEGENERATE CASES. QED

COR: SUPPOSE C_1, C_2 CUBIC, $C_1 \cap C_2 = \{p_1, .., p_9\}$

IF $\{p_1, .., p_8\} \subseteq C_3$ CUBIC

$$\Rightarrow \quad p_9 \in C_3 \text{ ALSO.}$$

GROUP LAW ON CUBIC

LET $C = Z(F) \subseteq \mathbb{P}^2_k$, $k \subseteq \mathbb{C}$, $\deg F = 3$, NON DEG, NONEMPTY.

ASSUME (1) F IRREDUCIBLE

(2) $\forall p \in C \; \exists! \; L \subseteq \mathbb{P}^2_k$ s.t. $F|L\,(p) = 0$ REPEATED.

(1) FIX ANY $O \in C$ (pt)

(2) FOR $A \in C$, LET $\bar{A}$ = 3RD PT OF INTERSECTION W/ LINE $\overline{OA}$

(3) $A, B \in C$, LET R = 3RD PT OF INT OF AB WITH C, DEFINE

$$A + B = \overline{R}$$

THEN $(C, +)$ IS AN ABELIAN GP.