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A. G CLASS 2

* REMINDER: CRITICAL READINGS

RECALL: AFFINE SPACE OVER THE FIELD k IS DENOTED $\mathbb{A}^n_k$.

$$\mathbb{A}^n_k = \{(x_1,..,x_n) \mid x_i \in k\}$$ AS A SET $= k^n$

LET $k[x_1,..,x_n]$ DENOTE THE POLYNOMIAL RING OVER k IN THE VARIABLES $x_1,\dots,x_n$. LET $S \subseteq k[x_1,..,x_n]$ BE ANY SET OF POLYNOMIALS. THEN

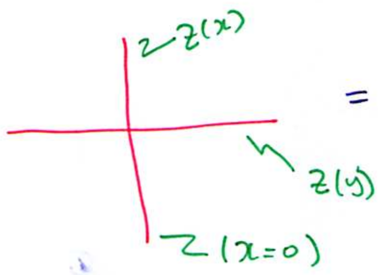
$$Z(S) = \{(a_1,..,a_n) \in \mathbb{A}^n_k \mid f(a_1,..,a_n) = 0 \ \forall f \in S\}$$

IS AN AFFINE ALGEBRAIC SET. A SUBSET $X \subseteq \mathbb{A}^n_k$ IS ALGEBRAIC IF $\exists\ S \subseteq k[x_1, x_n]$ SUCH THAT $X = Z(S)$.

DEF:

AN ALGEBRAIC SET $X \subseteq \mathbb{A}^n_k$ IS IRREDUCIBLE IF IT CANNOT BE EXPRESSED AS THE UNION OF TWO PROPER ALGEBRAIC SUBSETS.

EX: LET $f(x,y) = xy$, $f \in \mathbb{R}[x,y]$.

$Z(f) =$ 

$= Z(x) \cup Z(y)$

RECALL (READING): A CONIC IN $\mathbb{R}^2$ IS $Z(q)$, WHERE

$q \in \mathbb{R}[x,y]$ IS A POLYNOMIAL OF DEGREE 2 OR LESS.

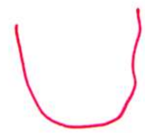
IE $q(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = 0$, $a, \ldots, f \in \mathbb{R}$.

TO GROUP: CAN YOU RECALL THESE?

CLASSIFICATION:

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ELLIPSE

(b) $y = mx^2$ PARABOLA



(c) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ HYPERBOLA

(d) $x^2 + y^2 = 0$ POINT

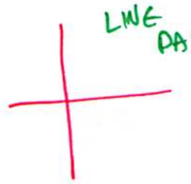
(e) $x^2 + y^2 = -1$ EMPTY SET $\emptyset$

(f) $x^2 = -1$ EMPTY SET $\emptyset$

(g) $0 = 1$ EMPTY SET $\emptyset$

(h) $x = 0$ LINE

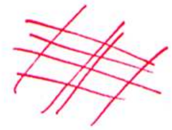
(i) $xy = 0$ LINE PAIR



(j) $x(x-1) = 0$ PARALLEL LINES

(k) $x^2 = 0$ "DOUBLE" LINE

(l) $0 = 0$ $\mathbb{R}^2$

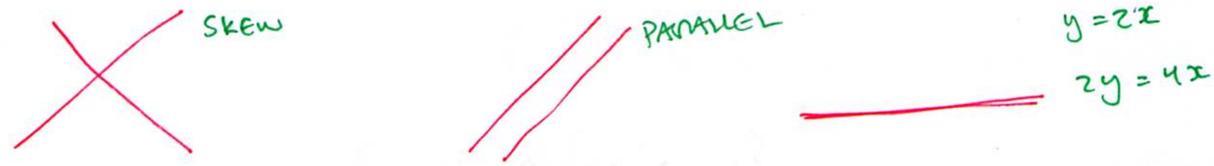


Q: WHICH ARE IRREDUCIBLE VARIETIES?

ENUMERATING INTRODUCTION TO PROJECTIVE SPACE

Q: WHAT IS THE INTERSECTION OF TWO LINES IN $\mathbb{R}^2$?

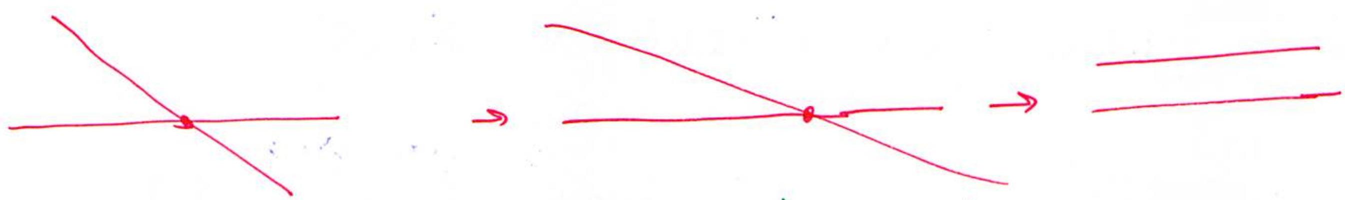
A:



1   0 ∞

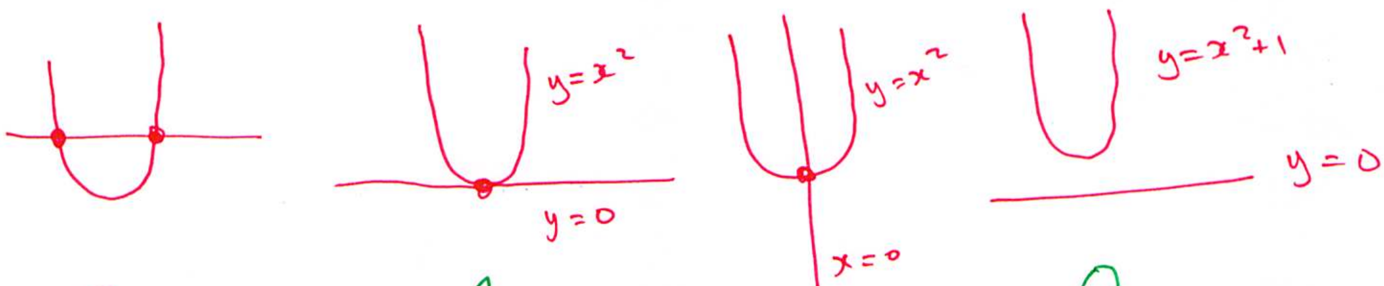
NOTE: THIS IS A PROBLEM! I WANT A NICE CLEAN ANSWER.

THE "PROBLEM" OF ∞ SOLUTIONS IS SOLVED BY REQUIRING DISTINCT LINES. BUT WHAT ABOUT NO SOLUTIONS.



POINT OF INTERSECTION "ESCAPES" TO ∞!

Q: WHAT IS THE INTERSECTION OF A LINE AND A CONIC IN $\mathbb{R}^2$?



2   1   1   0

NOTE: AGAIN, WE HAVE A PROBLEM!

SOLUTIONS: • COUNT THE ORDER OF INTERSECTION! THE NUMBER OF DERIVATIVES THAT AGREE IS IMPORTANT.

• ALSO COUNT INTERSECTIONS "AT ∞".

• ALSO NEED $\sqrt{-1}$! $\{y = x^2+1\} \cap \{y=0\}$

FINAL SOLUTION: $\mathbb{C}\mathbb{P}^2$ - THE COMPLEX PROJECTIVE PLANE!

$\boxed{\mathbb{C}\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}}.\ \text{3 DEFINITIONS.}}$

DEF 1: $\mathbb{C}\mathbb{P}^1 = \{(x,y) \in \mathbb{C}^2 \mid (x,y) \neq (0,0)\}/\sim$

$$(x,y) \sim \lambda(x,y) = (\lambda x, \lambda y) \quad \forall\ \lambda \neq 0 \in \mathbb{C}.$$

NOTATION: $(\mathbb{C}^2)^\times = \mathbb{C}^2 \setminus \{(0,0)\}$, $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

$\boxed{[(x,y)] = (x:y)}$ OR $[x:y]$.

EX:

$$(10^{10^{10^{10}}} : 0) = (1:0) = (5e^{3i} : 0)$$

$$(3i+4 : 3i+4) = (1:1)$$

DEF 2: WHAT IS $\mathbb{CP}^1$ AS A GEOMETRIC OBJECT?

LET $U \subseteq \mathbb{CP}^1$ BE DEFINED BY

$$U = \{(x:y) \in \mathbb{CP}^1 \mid x \neq 0\}.$$

CLAIM : $U \leftrightarrow \mathbb{C}$ — BIJECTION FOR NOW, ISOMORPHISM LATER

INVERSE MAPS

$$(x:y) \mapsto \frac{y}{x} \qquad \left((\lambda x : \lambda y) = \frac{\lambda y}{\lambda x} = \frac{y}{x} \right)$$

$$(1:z) \leftarrow z$$

SO $U \subseteq \mathbb{CP}^1$ IS A COPY OF THE COMPLEX PLANE.

WHAT IS LEFT OVER?

$$\mathbb{CP}^1 \setminus U = \{(x:y) \in \mathbb{CP}^1\} \setminus \{(x:y) \in \mathbb{CP}^1 \mid x \neq 0\}$$

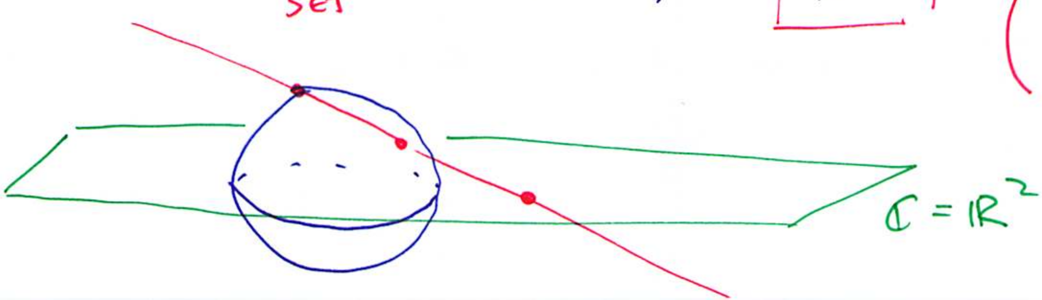
$$= \{(x:y) \in \mathbb{CP}^1 \mid x = 0\}$$

$$= \{(0:y) \in \mathbb{CP}^1\} = \{(0:1)\}.$$

SO $\mathbb{CP}^1 = U \cup \{(0:1)\}$

$$\underset{\text{SET}}{\overset{\text{AS A}}{=}} \mathbb{C} \cup \{\text{POINT}\} = \boxed{S^2}$$

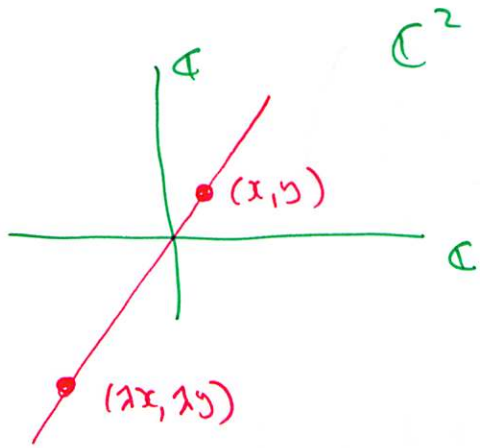
$\left(\{(0:1)\} = \infty ? \right.$
$\left. \text{WHY?} \right)$



DEF 3: AS A MODULI SPACE

$$\mathbb{C}\mathbb{P}^1 = \{\text{LINES THROUGH THE ORIGIN IN } \mathbb{C}^2\}$$

A SET WHOSE ELEMENTS ARE GEOMETRIC OBJECTS, USUALLY DEFINED UP TO SOME EQUIVALENCE RELATION.



DEF: LET k BE A FIELD. PROJECTIVE SPACE OF DIMENSION n OVER k IS

$$\mathbb{P}^n_k = \{\text{LINES THROUGH ORIGIN IN } \mathbb{A}^{n+1}_k\}$$

$$= \{(x_0, .., x_n) \in (\mathbb{A}^{n+1}_k)^\times\}/\sim$$

COUNTING CONVENTION

$$(x_0 : \dots : x_n) = \lambda(x_0 : \dots : x_n) = (\lambda x_0 : \lambda x_1 : \dots : \lambda x_n).$$

NOTE: $\mathbb{P}^n_k \neq S^{n+1}$ IN GENERAL (ONLY $k = \mathbb{C}, n = 1$)

Q: WHAT ABOUT ALGEBRAIC SUBSETS OF $\mathbb{P}^2_{\mathbb{C}}$?

GW: WHAT IS A CONIC IN $\mathbb{P}^2_{\mathbb{C}}$?

A: HOMOGENEOUS POLY'S CALLED FORMS.

NOTE: TWO DISTINCT LINES IN $\mathbb{P}^2_{\mathbb{C}}$ MEET IN A UNIQUE PT.

COUNTED W/ MULTIPLICITY, A CONIC AND A LINE MEET IN 2 POINTS IN $\mathbb{C}\mathbb{P}^2$.