DAMIAN KARP

A.G. CLASS ~~19~~ 20

DEF: LET $f \in k[x_1,..,x_n]$ BE AN IRREDUCIBLE NON-CONSTANT POLYNOMIAL,

AND

LET $V = Z(f) \subseteq \mathbb{A}^n$

AND SET $p = (a_1, \dots, a_n) \in V$. THE TANGENT SPACE TO V

AT p IS

$$T_p V = Z\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - a_i) \right) \subseteq \mathbb{A}^n.$$

NOTE: THIS IS THE EQ^N OF THE TANGENT PLANE THROUGH THE ORIGIN ~~FROM~~ M.V. CALC. HERE $\frac{\partial f}{\partial x_i}$ IS A FORMAL ALGEBRAIC OPERATION ~~NON-UCLESS~~ NOT REQUIRING CALC.

$$f(x_1,..,x_n) = \sum_{d_1+\cdots+d_n=d} c_{d_1 \cdots d_n} x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$$

$$\Rightarrow \frac{\partial f}{\partial x_i} = \sum_{d_1+\cdots+d_n=d} d_i \, c_{d_1 \cdots d_n} x_1^{d_1} x_2^{d_2} \cdots x_i^{d_i - 1} \cdots x_n^{d_n}$$

PROP: LET $L \subseteq \mathbb{A}^n$ BE A LINE THROUGH p. p IS A MULTIPLE ROOT OF $f|_L \iff L \subseteq T_pV$.

DEF: IF $L \subseteq T_pV$, $p \in L$, WE SAY L IS A TANGENT LINE TO V AT p.

PF! SINCE L IS A LINE IN $\mathbb{A}^n$ $\exists$ AN ISOMORPHISM

$$\mathbb{A}^1 \to L \subseteq \mathbb{A}^n$$
$$t \mapsto (a_1 + b_1 t, \ldots, a_n + b_n t).$$

THIS LINEAR ISOMORPHISM PARAMETERIZES L. NOTE

$$0 \mapsto (a_1, .., a_n) = p.$$

THEN $f|_L = f(a_1 + b_1 t, .., a_n + b_n t) = g(t)$. SINCE $p \in V = Z(f)$, WE KNOW $g(0) = f(p) = 0$. THEN

0 IS A MULT. ROOT OF g $\iff$ $\frac{\partial g}{\partial t}(0) = 0$

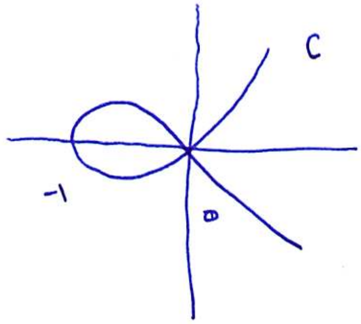
$$\iff \sum_1 b_i \frac{\partial f}{\partial x_i}(p) = 0 \iff L \subseteq T_pV.$$

QED

DEF: $p \in V \subseteq \mathbb{A}^n$ is a nonsingular point of V if $\frac{\partial f}{\partial x_i}(p) \neq 0$ for some i. Otherwise p is a singularity of V.

NOTE: If p is nonsingular, T_pV is an affine linear subspace of $\mathbb{A}^n$ of dimension $n-1$ (ie $T_pV \cong \mathbb{A}^{n-1}$). If p is singular, $T_pV = Z(0) = \mathbb{A}^n$. Singular points have a tangent space which is too large.

EX: $f(x,y) = y^2 - x^3 - x^2$. $\quad C = Z(f)$



$p_1 = (0,0) \qquad p_2 = (-1,0)$.

$f_x = -3x^2 - 2x \qquad f_y = 2y$

$$T_pV = Z\left(\sum_1 \partial_{x_i} f(p)(x_i - a_i)\right)$$

$$T_{p_1}C = Z(0\cdot x + 0\cdot y) = Z(0) = \mathbb{A}^2. \Rightarrow p_1 = (0,0) \text{ is a singular point of } C$$

$$T_{p_2}C = Z(-x + 0y) = Z(x) = y\text{-axis} \cong \mathbb{A}^1 \Rightarrow p_2 \text{ is a nonsingular pt of } C.$$

Also, note any line $L = Z(y - mx)$ through the origin satisfies $f|_L(p_1)$ is a repeated root.

THE BLOWUP IS ONE METHOD TO RESOLVE SINGULARITIES.

DEF: IF X IS A SINGULAR VARIETY (HAVING AT LEAST ONE SINGULAR POINT) A RESOLUTION OF SINGULARITIES FOR X IS A NONSINGULAR VARIETY Y SUCH THAT Y IS BIRATIONAL TO X.

DEF: THE BLOWUP OF $\mathbb{C}^2$ AT $p = (0,0)$ IS

$$Bl_p\,\mathbb{C}^2 = \left\{((x_1, x_2),(y_1 : y_2)) \in \mathbb{C}^2 \times \mathbb{P}^1 \;\middle|\; x_1 y_2 = x_2 y_1 \right\}$$

IT COMES EQUIPT WITH A PROJECTION

$$\pi : Bl_p\,\mathbb{C}^2 \to \mathbb{C}^2$$

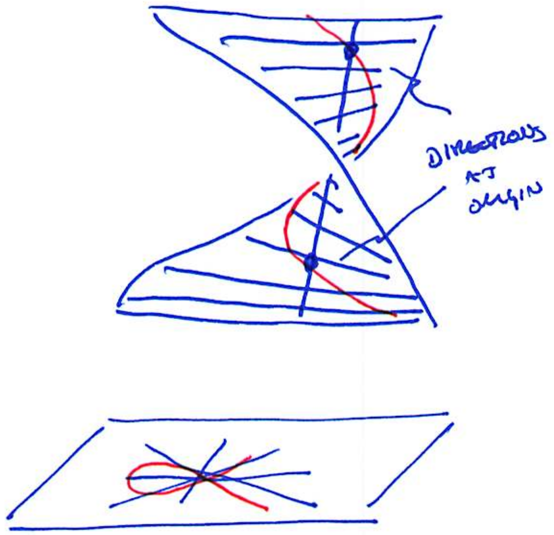
$$\pi(x,y) = x.$$

FOR $(x_1, x_2) \neq (0,0)$ $\quad \pi^{-1}(x_1, x_2) = ((x_1, x_2),(x_1 : x_2))$

THUS $\pi : Bl_p\,\mathbb{C}^2 \setminus \pi^{-1}\{p\} \to \mathbb{C}^2 \setminus (0,0)$ IS AN ISOMORPHISM.

AND $Bl_p\,\mathbb{C}^2$ IS BIRATIONAL TO $\mathbb{C}^2$.

ALSO $\pi^{-1}(0,0) = \{(0,0),(y_1 : y_2) \mid (y_1 : y_2) \in \mathbb{P}^1\} = \vec{0} \times \mathbb{P}^1 \cong \mathbb{P}^1$



WE WANT TO IDENTIFY

$$\tilde{C} = \overline{\pi^{-1}(C \setminus \vec{0})} \subseteq \mathbb{C}^2 \times \mathbb{P}^1$$

LET $(x,y) \in C$, $(x,y) \neq (0,0)$.

THEN $\pi^{-1}(x,y) = ((x,y),(u:v))$

$$y^2 = x^3 + x^2, \quad xv = yu.$$

EITHER $u=0$ OR $u \neq 0$.

IF $u \neq 0$ SET $(u:v) = (1:v)$ WLOG. THEN $y = xv$. HENCE

$$(xv)^2 = x^2(x+1) \Rightarrow x^2(x+1-v^2) = 0$$

SO $x=0$ OR $x = v^2 - 1$. IF $x=0$ THEN $y=0$. THUS $x \neq 0$,

AND $x = v^2-1$. SO $v \neq 1,-1$. AND $y = v(v^2-1)$.

WE HAVE $$\{(v^2-1, v(v^2-1), (1:v))\}_{v \neq 1,-1} = \pi^{-1}(C \setminus \vec{0}) \cap U = \{u \neq 0\}.$$

IF $u=0$ THEN $v \neq 0$. WLOG $(u:v) = (u:1)$ SO $x = yu$.

$$\Rightarrow y^2 = (yu)^3 + (yu)^2 \Rightarrow y^2(yu^3 + u^2 - 1) = 0 \Rightarrow$$

$y=0$ OR $yu^3 = 1-u^2$. $y=0 \Rightarrow x = 0u = 0$. SO $y \neq 0$, $u \neq 1,-1$.

ALSO, $u=0 \Rightarrow x=0 \Rightarrow y=0$ SO $u \neq 0$. THEN

$$y = \frac{1-u^2}{u^3} \qquad x = \frac{1-u^2}{u^2} \Rightarrow y = \frac{1}{u}\left(\left(\frac{1}{u}\right)^2 - 1\right), \quad x = \left(\frac{1}{u}\right)^2 - 1$$

$$\left\{ \left(\left(\tfrac{1}{u}\right)^2 - 1,\ \tfrac{1}{u}\left(\left(\tfrac{1}{u}\right)^2 - 1\right) \right),\ (u:1) \;\middle|\; u \neq 0, 1, -1 \right\}.$$

SAME PARAMETERIZATIONS. SO WE HAVE A MAP

$$\mathbb{A}^1 \xrightarrow{\varphi} \tilde{C}$$

$$t \longmapsto \left((t^2-1,\ t(t^2-1)),\ (1:t) \right)$$

WHEN $t = 1, -1$ WE RECOVER THE TWO MISSING POINTS

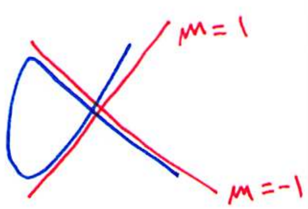
$$((0,0),(1:1)) \quad \text{AND} \quad ((0,0),(1:-1)).$$

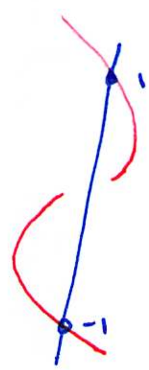
THIS COVERS ALL CASES WHERE $u = 1$, INCLUDING $(x,y) = (0,0)$.

WHAT IF $(x,y) = (0,0)$ AND $u = 0$? THEN $v = 1$ AND WE HAVE

$$((0,0),(0:1)).$$

NOTE
$$\left.\begin{aligned} \nabla\varphi(1) &= (2,2) \parallel (1,1) \\ \nabla\varphi(-1) &= (-2,2) \parallel (1,-1) \end{aligned}\right\} \text{ MISSING POINTS ARE DIRECTIONS AT } (0,0)!$$





3RD INTERSECTION W/ $\mathbb{P}^1$ OFF AT ∞!