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A.G. CLASS 22

$k = \bar{k}$

PROP: LET $f \in k[x_1, .., x_n]$ BE IRREDUCIBLE AND NONCONSTANT.

LET $V = Z(f) \subseteq \mathbb{A}^n$. DEFINE

$$V_{SING} = \{ \text{SINGULAR POINTS OF } V \} \quad \text{AND}$$

$$V_{NONSING} = V \setminus V_{SING}.$$

THEN $V_{NONSING}$ IS AN OPEN DENSE SUBSET OF V.

PF: WE FIRST SHOW $V_{NONSING}$ IS OPEN BY SHOWING

V_{SING} IS CLOSED. $p \in V = Z(f)$ IS SINGULAR

IFF $\frac{\partial f}{\partial x_i}(p) = 0$ FOR ALL $i = 1, .., n$. THUS

$$V_{SING} = Z\left(f, \frac{\partial f}{\partial x_1}, .., \frac{\partial f}{\partial x_n}\right)$$

HENCE V_{SING} IS CLOSED AND $V_{NONSING}$ IS OPEN.

$V = Z(f)$ IS A VARIETY (IRRED.) B/C f IS IRRED.

THUS NONEMPTY OPEN SETS ARE DENSE, SO WE ONLY NEED

TO SHOW THAT $V_{NONSING}$ IS NONEMPTY.

SUPPOSE $V_{NS} = \emptyset$. THEN $V = V_{SING}$

AND EVERY POINT OF V IS A SINGULAR POINT. THUS

$$\frac{\partial f}{\partial x_i}(p) = 0 \quad \forall\, p \in V \quad \text{AND} \quad \forall\, i = 1, \ldots, n.$$

THEN $\frac{\partial f}{\partial x_i} \in I(Z(f)) = \sqrt{(f)} = (f)$ (AS f IS IRRED $\Rightarrow (f)$ PRIME $\Rightarrow \sqrt{(f)} = (f)$).

SO $\frac{\partial f}{\partial x_i} \in (f)$. THUS $\exists\, g \in k[x_1, \ldots, x_n]$

SUCH THAT $fg = \frac{\partial f}{\partial x_i}$. BUT

$$\deg_{x_i}(fg) \geq \deg_{x_i}(f) > \deg_{x_i} \frac{\partial f}{\partial x_i}$$

UNLESS f DOESN'T DEPEND ON x_i. BUT f IS

NONCONSTANT SO MUST DEPEND ON SOME x_i. $\lightning$ QED

(NEED SLIGHTLY MORE DELICATE ARGUMENTS FOR $\mathrm{CHAR}(k) = p$.
INSTEAD OF NO DEPENDENCE, HAVE x_i^p FOR ALL x_i
ON WHICH f DEPENDS. THEN $f = g^p$ SOME $g \in A$,
CONTRADICTING IRREDUCIBILITY OF f.)

Q: WHAT ABOUT THE GENERAL (NON HYPERSURFACE) CASE?.

DEF: LET $V \subseteq \mathbb{A}^n$, $p = (a_1, \ldots, a_n) \in V$. FOR ANY $f \in k[x_1, \ldots, x_n]$, DEFINE THE FIRST ORDER PART OF f AT p TO BE

$$f_p^{(1)} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - a_i).$$

THE TANGENT SPACE TO V AT p IS DEFINED BY

$$T_p V = \bigcap_{f \in I(V)} Z(f_p^{(1)}) \subseteq \mathbb{A}^n.$$

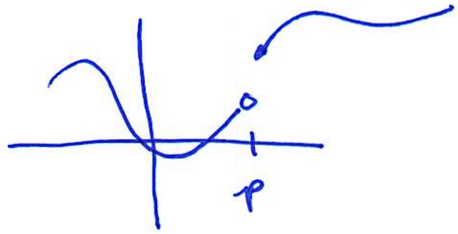
EX: LET $V = Z(y, z) \subseteq \mathbb{A}^3$ $p = (1, 0, 0)$.

FIND $T_p V$.

DEF: LET X BE A TOP. SPACE. A FUNCTION $f : X \to \mathbb{R}$ IS CALLED UPPER SEMI-CONTINUOUS AT $p \in X$ IF $\forall \varepsilon > 0$ $\exists$ NBHD U OF p SUCH THAT

$$f(x) \leq f(p) + \varepsilon \quad \forall x \in U.$$

EX:



NOTE: COMPLIMENT IS CLOSED.

PROP: THE FUNCTION $V \to \mathbb{N}$, $p \mapsto \dim T_p V$

IS UPPER SEMICONTINUOUS, HENCE FOR ANY $r \in \mathbb{N}$,

$$S(r) = \{ p \in V \mid \dim T_p V \geq r \} \subseteq V$$

IS CLOSED.

GENERAL FACT: ANY UPPER S.C. FUNCTION $X \to \mathbb{R}$ IS LOCALLY CONSTANT ON A DENSE OPEN SUBSET OF X.

PF: LET $I(V) = (f_1, \dots, f_m)$. THEN

$$T_p V = \bigcap_{f \in I(V)} Z(f_p^{(1)}) = \bigcap_{i=1}^{m} Z(f_{i\,p}^{(1)})$$

$$p \in S(r) \Leftrightarrow \text{RANK} \left(\frac{\partial f_i}{\partial x_j}(p) \right) \leq n - r$$

$\Leftrightarrow$ EVERY $(n-r+1) \times (n-r+1)$ MINOR VANISHES.

THIS IS A POLYNOMIAL (DETERMINANT) CONDITION. QED

OUR DEF: $\exists\, r \in \mathbb{N}$ AND A DENSE OPEN SET $V_0 \subseteq V$ s.t.

$$\dim T_pV = \begin{cases} r, & p \in V_0 \\ \geq r & p \in V \end{cases}$$

AND r IS CALLED THE DIMENSION OF V. $p \in V$ IS NONSINGULAR IF $\dim T_pV = r$ AND SINGULAR IF $\dim T_pV > r$.

PF: LET $r = \min_{p \in V} \{\dim T_pV\}$. THEN

~~$S(r+1) \neq \emptyset$~~ $S(r) = V$ $\quad S(r+1) \subsetneq V$.

$S(r) \setminus S(r+1) = \{ p \in V \mid \dim T_pV = r \}$ OPEN, NONEMPTY.

QED.

DEF: LET $k \subseteq K$ BE A FIELD EXTENSION. THEN $\alpha_1, \ldots, \alpha_m \in K$ ARE ALGEBRAICALLY INDEPENDENT IF THEY SATISFY NO NONTRIVIAL POLYNOMIAL EQUATION W/ COEFFICIENTS IN k. THEY SPAN THE TRANSCENDENTAL PART OF K IF $K / k(\alpha_1, \ldots, \alpha_m)$ IS ALGEBRAIC. THEY FORM A TRANSCENDENCE BASIS IF THEY ARE ALG. INDEP. AND SPAN.

THM: A TRANSCENDENCE BASIS IS A MAX'L ALG. INDEP. SET) AND A MIN SPANNING SET, AND ANY TWO BASES HAVE THE SAME # OF ELTS.

DEF: THE # OF ELTS IN A TRANSCENDENCE BASIS IS CALLED THE TRANSCENDENCE DEGREE OF THE EXTENSION $k \subseteq K$.

THM! $\dim V = \operatorname{tr\,deg} k(V)$

THM:

(a) $$(T_p V)^* = m_p / m_p^2$$

(b) IF $f(p) \neq 0$ FOR ~~ALL~~ $V_f \subseteq V$, $f \in k[V]$, THEN

$$T_p(V_f) \xrightarrow{\sim} T_p V \qquad \left(V_f = V \setminus Z(f).\right)$$