DAGAN KARP

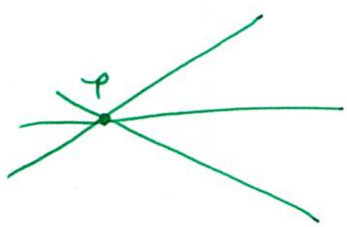
AG CLASS 23

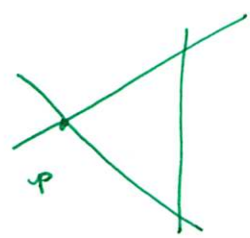
LET $S = Z(f) \subseteq \mathbb{P}^3$ BE A NONSINGULAR CUBIC SURFACE, AND $f(x,y,z,t)$ ITS DEFINING 3-FORM.

THM: THERE ARE 27 DISTINCT LINES ON S.

WE WILL PROVE THIS THM IN SEVERAL STAGES.

PROP: (a) THERE ARE AT MOST 3 LINES IN S THROUGH ANY POINT $p \in S$. MOREOVER, IF THERE ARE 2 OR 3, THEY ARE COPLANAR.





(b) EVERY PLANE $H \subseteq \mathbb{P}^3$ INTERSECTS S IN ONE OF THE FOLLOWING.

(i) AN IRREDUCIBLE CUBIC CURVE 

(ii) A CONIC PLUS A LINE 

(iii) 3 DISTINCT LINES 

PF: (a) LET $\ell \subseteq S$ BE A LINE. FOR $p \in \ell$,

$$\ell = T_p \ell \subseteq T_p S$$

THUS ALL LINES THROUGH $p \in S$ ARE CONTAINED IN $T_p S$.

THERE ARE AT MOST 3 LINES IN $T_p S \cap S$ BY (b).

(b) WE MUST SHOW THAT $H \cap S$ CONTAINS NO MULTIPLE LINE.

BY CHANGE OF COORDINATES, $H = Z(t)$, $\ell = Z(z,t) \subseteq H$.

IF $\ell \subseteq H \cap S$ IS A MULTIPLE LINE, THEN f

MUST BE OF THE FORM

$$f = z^2 \cdot g(x,y,z,t) + t\,h(x,y,z,t)$$

WHERE g IS LINEAR AND h IS QUADRATIC. BUT THEN

$S = Z(f)$ IS SINGULAR AT $Z(h,z,t) \neq \emptyset$ ROOTS OF h ON ℓ.

QED

PROP: THERE EXISTS AT LEAST ONE LINE ℓ ON S.

METHOD 1: DIMENSION COUNT.

A LINE IN $\mathbb{P}^3$ IS GIVEN BY $ax+by+cz+dt=0$.

FOR ANY $\lambda \in k^\times$, $\lambda ax + \lambda by + \lambda cz + \lambda dt = 0$ IS THE SAME LINE. THUS

$$\{\text{LINES IN } \mathbb{P}^3\} = \{(a,b,c,d) \in \mathbb{A}^4 \setminus \{0\}\}/\sim \; = \mathbb{P}^3$$

$\ell \subseteq S \iff f|_\ell = 0$ CUBIC IN 3 VARIABLES VANISHES...

DIRECT GEOMETRY: FOR ANY $p \in S$, $T_pS \cap S = C$ IS A SINGULAR PLANE CUBIC. IF C IS REDUCIBLE, WE ARE DONE. OTHERWISE, C IS A NODAL OR CUSPIDAL CUBIC. CHOOSE COORDINATES SO THAT $T_pS = Z(t)$, $p = (0:0:1:0)$,

$$C = Z(xyz - x^3 - y^3) \text{ OR } (y^3 - x^2 z).$$

WE PROVE THE CUSPIDAL CASE, BUT NODAL GET THE SAME IDEAS.

SO, ASSUME $S = Z(f)$ WITH $f(x,y,z,t) = x^2 z - y^3 + gt$ WHERE $g(x,y,z,t)$ IS A QUADRATIC FORM. SINCE S IS NONSINGULAR AT p, $g(0:0:1:0) \neq 0$ WLOG $g(p) = 1$.

CONSIDER THE POINT $p_\alpha = (1 : \alpha : \alpha^3 : 0) \in C \subseteq S'$. ANY LINE

$\ell \subseteq \mathbb{P}^3$ THROUGH p_α HAS $\ell \cap H = Z(x) = q = (0 : y : z : t)$.

THE CONDITIONS FOR THE LINE $\ell_{p_\alpha q} \subseteq S'$ CAN BE EXPRESSED

IN TERMS OF α, q. $f(\lambda p_\alpha + \mu q)$ GIVES

$$\ell_{p_\alpha q} \subseteq S' \iff A(y,z,t) = B(y,z,t) = C(y,z,t) = 0$$

WHERE A, B, C ARE FORMS OF DEG $1, 2, 3$ DEPENDING ON α.

CLAIM: $\exists$ A RESULTANT POLYNOMIAL $R_{27}(\alpha)$ WHICH IS MONIC

OF DEGREE 27 IN α s.t.

$$R(\alpha) = 0 \iff A = B = C = 0 \text{ HAVE A COMMON ZERO IN } \mathbb{P}^2$$
$$(\eta, \xi, \tau)$$

PF OF CLAIM: SAME SYLVESTER'S DET WE'VE SEEN BEFORE.

DETAILS IN BOOK.

FOR EVERY ROOT α OF R, $\exists$ $q = (0 : \eta : \xi : \tau)$ IN H

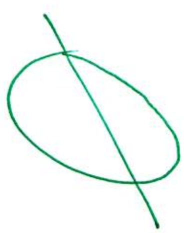
WITH $\ell_{p_\alpha q} \subseteq S'$. QED.

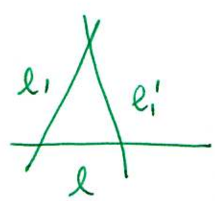
PROP: GIVEN $\ell \subseteq S'$, $\exists$ EXACTLY 5 PAIRS OF LINES (ℓ_i, ℓ_i') OF S' MEETING ℓ SUCH THAT

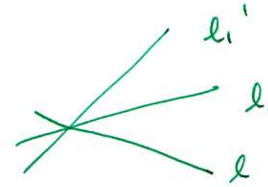
(1) $\ell \cup \ell_i \cup \ell_i'$ IS COPLANAR FOR $i = 1, \dots, 5$

(2) $(\ell_i \cup \ell_i') \cap (\ell_j \cap \ell_j') = \emptyset$ FOR $i \neq j$

PF: IF H IS A PLANE IN $\mathbb{P}^3$ WITH $\ell \subseteq H$, THEN $f|_H$ IS DIVISIBLE BY ℓ'S EQ, SO $H \cap S' = \ell \cup$ CONIC.







WTS $\exists$ EXACTLY 5 PLANES $\ell \subseteq H_i$ FOR WHICH SINGULAR OCCURS

TO SHOW (1)., (2) FOLLOWS FROM FACT THAT LINES IN DIFFERENT PLANES ARE DISJOINT, PART (a) PROVED EARLIER.