DAGAN KARP

A.G. CLASS 3

RECALL! $\mathbb{CP}^1 = \{(x,y) \in (\mathbb{C}^2)^\times\}/\sim$, WHERE $(x:y) = (\lambda x : \lambda y)\ \forall \lambda \in \mathbb{C}^\times$

$= \{\text{LINES THROUGH } \vec{0} \text{ IN } \mathbb{C}^2\}$

DEF: $\mathbb{CP}^n = \{(x_0, x_1, \dots, x_n) \in (\mathbb{C}^{n+1})^\times\}/\sim$,

WHERE $(x_0 : x_1 : \dots : x_n) = (\lambda x_0 : \lambda x_1 : \dots : \lambda x_n)\ \forall \lambda \in \mathbb{C}^\times$.

IN GENERAL, FOR A FIELD k, PROJECTIVE n-SPACE OVER k IS

$\mathbb{P}^n_k = \{(x_0, \dots, x_n) \in (k^{n+1})^\times\}/\sim$, WHERE

$(x_0 : \dots : x_n) = (\lambda x_0 : \dots : \lambda x_n)\ \forall \lambda \in k^\times$.

AND $\mathbb{P}^n_k = \{\text{LINES THROUGH THE ORIGIN IN } \mathbb{A}^{n+1}_k\}$

MODULI SPACE: A GEOMETRIC OBJECT WHOSE POINTS CORRESPOND TO (EQUIVALENCE CLASSES OF) GEOMETRIC OBJECTS.

DEF:

LET $U_i = \{(x_0 : \dots : x_n) \in \mathbb{P}^n_k \mid x_i \neq 0\}$. THESE ARE CALLED THE STANDARD AFFINE OPEN SUBSETS OF $\mathbb{P}^n_k$.

NOTE: $U_i \xleftrightarrow{\ \sim\ } \mathbb{A}^n_k$

BIJECTION

EG: $U_0 \longmapsto \mathbb{A}^n_k \qquad (x_0 : \dots : x_n) \mapsto (x_1/x_0, \dots, x_n/x_0)$

$U_0 \longleftarrow \mathbb{A}^n_k \qquad (1 : z_1 : \dots : z_n) \leftarrow (z_1, \dots, z_n)$

(SO $\mathbb{P}^n_k$ IS COVERED BY AFFINE SETS.)

EX: $\mathbb{P}^2_{\mathbb{C}} = \mathbb{CP}^2$. WHAT ABOUT CONICS IN $\mathbb{P}^2$?

$$f(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$$

$$\left(f = x^2 + 3y. \quad f(1:1) = f(2:2)\,? \right)$$

SOLUTIONS: HOMOGENEOUS POLY'S HAVE WELL DEFINED ZERO SETS.

CALLED FORMS

DEF: A CONIC IN $\mathbb{P}^2_k$ IS THE ZEROS OF A QUADRATIC FORM, IE A HOM. POLY OF DEGREE 2.

$$Q(x:y:z) = ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2$$

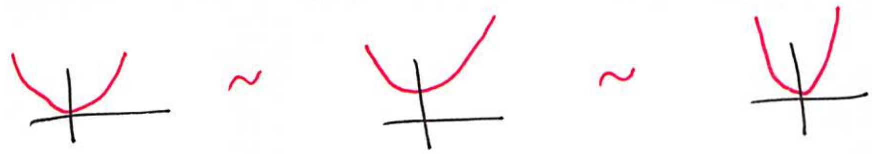
A LINE IN $\mathbb{P}^2_k$ IS $Z(f)$, WHERE f IS HOMOG. OF DEGREE 1, IE A LINEAR FORM, IE

$$f(x,y,z) = ax + by + cz$$

HW: TWO DISTINCT LINES IN $\mathbb{CP}^2$ MEET IN A UNIQUE PT

Q: WHAT IS THE CLASSIFICATION OF CONICS IN $\mathbb{P}^2_{\mathbb{R}}$?

IN $\mathbb{R}^2$, WHY ARE $y = x^2$ AND $y = x^2 + 1$ AND $y = 4x^2$ "THE SAME"?



WE MAY TRANSFORM ONE TO ANOTHER VIA A CHANGE OF COORDINATES:

$$(x,y) \mapsto (x, y+1) \qquad (2x, y)$$

WHAT IS A CHANGE OF COORD'S?

DEF: AN AFFINE TRANSFORMATION ON $\mathbb{R}^2$ IS A MAP OF THE FORM $T(\vec{x}) = A\vec{x} + B$,

A IS INV. 2×2, B TRANSLATION VECTOR

IF A IS ORTHOGONAL, THEN T IS EUCLIDEAN.

GW: (1) WHAT IS AN ORTHOGONAL MATRIX? WHAT DOES IT DO GEOMETRICALLY?

(2) CAN YOU FIND TRANSFORMATIONS TRANSFORMING THE ABOVE PARABOLAS?

CLAIM: EVERY NONDEGENERATE CONIC CAN BE REDUCED TO THE STANDARD PARABOLA, HYPERBOLA, ELLIPSE VIA A EUCLIDEAN TRANSF. EVERY CONIC CAN BE REDUCED TO STANDARD FORMS VIA AN AFFINE TRANSFORMATION. EXERCISE

Q: WHAT TRANSFORMATIONS YIELD PROJECTIVE GEOMETRY?

DEF: A PROJECTIVE TRANSFORMATION, OR PROJECTIVITY IS A MAP $T: \mathbb{P}^2 \to \mathbb{P}^2$ OF THE FORM

$$T(\vec{x}) = M\vec{x}$$

FOR AN INVERTIBLE 3×3 MATRIX M.

SAME FOR $\mathbb{P}^n_{\mathbb{R}} \to \mathbb{P}^n_{\mathbb{R}}$

CONICS IN $\mathbb{RP}^2$:

(1) NONDEGENERATE: $x^2+y^2-z^2=0$

(2) EMPTY: $x^2+y^2+z^2=0$

(3) LINE PAIR: $x^2-y^2=0$

(4) POINT: $x^2+y^2=0$

(5) DOUBLE LINE: $x^2=0$

DEF: LET $F(x,y)$ BE A DEGREE d FORM OVER k.

THE ASSOCIATED NON-HOMOGENEOUS POLY TO F IS

$$f(x) = F(x,1).$$

IE

$$F(x,y) = a_d x^d + a_{d-1}x^{d-1}y + \cdots + a_0 y^d$$

$$f(x) = a_d x^d + \cdots + a_0$$

SIMILARLY, FOR A POLY $g(x)$ OF degree d,

THE HOMOGENIZATION IS

$$G(x,y) = y^d g(x/y)$$

WORKS FOR ANY # OF VARIABLES

$$g(x) = a_d x^d + \cdots + a_0$$

$$G(x,y) = \left(a_d\left(\frac{x^d}{y^d}\right) + \cdots + a_1\frac{x}{y} + a_0\right)y^d = a_d x^d + \cdots + a_1 x y^{d-1} + a_0 y^d.$$