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A.G. CLASS 4

<u>RECALL:</u> IF f IS A <u>FORM</u> OF DEGREE d (<u>HOMOGENEOUS POLYNOMIAL</u>) IN $(x_0,..,x_n)$, THEN

$$f(x_0,..,x_n)=0 \iff \underbrace{f(\lambda x_0,..,\lambda x_n)}_{\lambda^d f(x_0,..,x_n)}=0$$

THUS THE ZEROS OF f, OR <u>ZERO LOCUS OF f</u>, IS WELL DEFINED ON $\mathbb{P}^n_k$:

$$Z(f)=\{(x_0:..:x_n)\in\mathbb{P}^n \mid f(x_0:..:x_n)=0\}.$$

SIMILARLY, IF $\{f_\alpha\}_{\alpha\in I}$ IS A COLLECTION OF <u>FORMS</u> IN $(x_0,..,x_n)$, NOT NEC. OF SAME DEGREE.
THEN

$$Z(\{f_\alpha\})=\{(a_0:...:a_n)\in\mathbb{P}^n_k \mid f_\alpha(a_0,..,a_n)=0\ \forall\ \alpha\in I\}.$$

<u>DEF:</u> A SUBSET $X\subset\mathbb{P}^n_k$ IS <u>ALGEBRAIC</u> IF $\exists$ A COLLECTION OF FORMS $\{f_\alpha\}_{\alpha\in I}$ SUCH THAT

$$X=Z(\{f_\alpha\}).$$

AN ALGEBRAIC SUBSET OF $\mathbb{P}^n_k$ IS <u>IRREDUCIBLE</u> IF IT CANNOT BE EXPRESSED AS A UNION OF TWO PROPER ALGEBRAIC SUBSETS.

DEF: AN IRREDUCIBLE ALGEBRAIC SUBSET OF $\mathbb{P}^n_k$ IS CALLED A <u>PROJECTIVE ALGEBRAIC VARIETY</u>.

EX: RECALL CONICS IN $\mathbb{RP}^2$ FROM LAST LECTURE = CLASS 3. *

WHICH ARE VARIETIES?

CLAIM: (1) THE EMPTY SET IS AN ~~AFFINE VARIETY~~ ALGEBRAIC SUBSET OF $\mathbb{A}^n$

(2) $\mathbb{A}^n_k$ IS AN ALGEBRAIC SUBSET OF $\mathbb{A}^n_k$

(3) THE INTERSECTION OF ANY COLLECTION OF ALGEBRAIC SUBSETS OF $\mathbb{A}^n_k$ IS AN ALGEBRAIC SUBSET OF $\mathbb{A}^n_k$

(4) THE UNION OF A FINITE COLLECTION OF ALGEBRAIC SUBSETS OF $\mathbb{A}^n_k$ IS AN ALGEBRAIC SUBSET OF $\mathbb{A}^n_k$.

SIMILARLY FOR $\mathbb{P}^n_k$.

PROOF: <u>GROUPWORK</u>. (15 MIN OR AS LONG AS NECESSARY W/ STUDENT PRESENTATIONS + DISCUSSION)

$\boxed{\text{INTRODUCTION TO TOPOLOGY}}$

BIG IDEA: TOPOLOGY IS THE BRANCH OF MATHEMATICS WHICH STUDIES THE MOST GENERAL NOTION OF <u>OPEN</u> AND <u>CLOSED</u> SETS. NOT REQUIRING A NOTION OF DISTANCE OR ANGLE (CURVATURE), IT IS OFTEN DESCRIBED AS RUBBER GEOMETRY.

DEF: LET X BE A SET. A COLLECTION $\mathcal{T}$ OF SUBSETS OF X IS CALLED A TOPOLOGY ON X IF

(1) $\emptyset \in \mathcal{T}$

(2) $X \in \mathcal{T}$

(3) IF $U_\alpha \in \mathcal{T}$ FOR ALL $\alpha \in I$ (ANY INDEXING SET I), THEN $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

(4) IF $U_1, .., U_n \in \mathcal{T}$, THEN $U_1 \cap \dots \cap U_n \in \mathcal{T}$.

IN THAT CASE, ELEMENTS $U \in \mathcal{T}$ ARE CALLED OPEN SETS.
THE PAIR $(X, \mathcal{T})$ IS CALLED A TOPOLOGICAL SPACE.

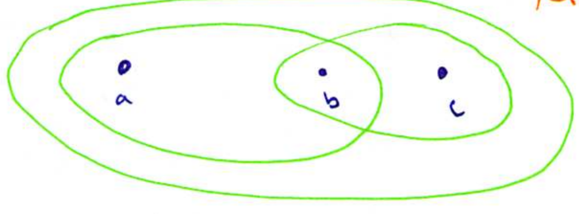
DEF: A SET $V \subseteq X$ IS CLOSED IN THE TOPOLOGY $\mathcal{T}$ IF ITS COMPLEMENT $X \setminus V$ IS OPEN, IE $X \setminus V \in \mathcal{T}$.

EX: $\mathbb{R}$, METRIC TOPOLOGY.

GW Q! WHY ONLY FINITE INTERSECTIONS? — THINK, PAIR, SHARE

EX: $X = \{a, b, c\}$

TOPOLOGY?



GROUPWORK: FIND MORE TOPOLOGIES ON X. (5 MIN)

DEF: LET X BE ALGEBRAIC. THE ZARISKI TOPOLOGY ON X IS DEFINED BY DECLARING ALGEBRAIC SUBSETS OF X TO BE CLOSED.

NOTE: ALL VARIETIES & ALG. SETS ARE ALSO TOPOLOGICAL SPACES! TUVOETLS

Q: WHAT ARE THE OPEN SETS IN $\mathbb{R} = \mathbb{A}^1_{\mathbb{R}}$?

(IN THE ZARISKI TOPOLOGY) (GROUPWORK)

DEF: AN OPEN SUBSET OF AN ALGEBRAIC* VARIETY IS CALLED A QUASI-AFFINE OR QUASI-PROJECTIVE VARIETY.

EX: $U_i = \{ (x_0 : \ldots : x_n) \in \mathbb{P}^n_k \mid x_i \neq 0 \}$ IS A QUASI-PROJECTIVE VARIETY AS $U_i = \mathbb{P}^n_k \setminus Z(x_i)$. HENCE U_i ARE OPEN, AND THE NAME STANDARD OPEN SETS IS APPROPRIATE.