DAMIAN KAMP

ALGEBRAIC GEOMETRY CLASS 5

RECALL: LAST TIME WE SAW AN INTRODUCTION TO TOPOLOGICAL SPACES. (GO OVER THIS MATERIAL - IT WAS TOO QUICK LAST TIME AND WE DIDN'T COVER ALL THE MATERIAL.)

NOTE: DISCUSS SYNTHESIS, ANALYSIS, AND HEURISTICS.

DEF: LET $Y \subseteq \mathbb{A}^n$ BE ANY SUBSET. THE IDEAL OF Y IN $A = k[x_1, .., x_n]$ IS DEFINED TO BE

$$I(Y) = \{ f \in A \mid f(p) = 0 \ \forall \ p \in Y \}.$$

CLAIM: $I(Y)$ IS AN IDEAL. (THINK PAIR SHARE ~2 MIN)

LET $p \in Y$, $f, g \in I(Y)$. THEN $f(p) - g(p) = 0 - 0 = 0 \Rightarrow f - g \in I(Y)$

THUS $I(Y) \leq A$. LET $h \in A$ BE ANY POLYNOMIAL THEN

$$f(p)h(p) = 0 \cdot h(p) = 0 \Rightarrow fh \in I(Y).$$

THEREFORE $I(Y)$ IS AN IDEAL OF A.

NOTE:

$$\left\{ \begin{array}{c} \text{ALGEBRAIC} \\ \text{SUBSETS OF} \\ \mathbb{A}^n \end{array} \right\} \underset{Z}{\overset{I}{\rightleftarrows}} \left\{ \begin{array}{c} \text{IDEALS} \\ \text{OF } A \end{array} \right\}$$

NOTE: LET'S INSPECT IDEALS MORE CLOSELY. LET $S \subseteq A$ BE ANY COLLECTION OF POLYNOMIALS. THE IDEAL GENERATED BY S IS

AUDIENCE GENERATED

$$(S) = \{\text{LINEAR COMBINATIONS OF ELT'S OF } S \text{ W/ COEFFICIENTS IN } A\}$$

CLAIM: $Z(S) = Z((S))$

PF: LET $p \in Z(S)$. THEN $f(p) = 0 \ \forall f \in S$.

LET $h \in (S)$. THEN $\exists f_1, .., f_n \in S$ AND $g_1, .., g_n \in A$ SUCH THAT

$$h = f_1 g_1 + f_2 g_2 + \cdots + f_n g_n.$$

THEN $h(p) = \sum_{i=1}^{n} f_i(p) g_i(p) = \sum_{i=1}^{n} 0 \, g_i(p) = 0.$

THUS $p \in Z((S))$ AND HENCE $Z(S) \subseteq Z((S))$.

CONVERSELY, SUPPOSE $p \in Z((S))$. THEN $h(p) = 0 \ \forall h \in (S)$.

LET $f \in S$. THEN $f \in (S)$. THUS $f(p) = 0$. THEREFORE

$$Z((S)) \subseteq Z(S).$$

HENCE $Z(S) = Z((S))$.

REMARK: IT IS SUFFICIENT TO STUDY IDEALS OF A (NOT ARBITRARY COLLECTIONS OF POLY'S)

RECALL: LET R BE A COMMUTATIVE RING W/ 1. THEN $R[x]$ THE RING OF POLYNOMIALS IN x W/ COEFFICIENTS IN R IS A PID IF AND ONLY IF R IS A FIELD.

IN PARTICULAR, FOR ANY FIELD k, $k[x]$ IS A PID.

EMMY NOETHER 1882-1935

DEF: LET R BE COMM. W/ 1. R IS A NOETHERIAN RING IF THE IDEALS OF R SATISFY THE ASCENDING CHAIN CONDITION, IE FOR EVERY CHAIN OF IDEALS

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

THERE EXISTS $n \in \mathbb{N}$ SUCH THAT $I_n = I_{n+1} = I_{n+2} = \cdots$.

THM! R IS NOETHERIAN $\Leftrightarrow$ EVERY IDEAL OF R IS FINITELY GENERATED

COR: $k[x]$ IS NOETHERIAN. (b/c $1 \neq \infty$)

DAVID HILBERT 1862-1943

HILBERT'S BASIS THM!

IF R IS A NOETHERIAN RING, THEN $R[x]$ IS ALSO A NOETHERIAN RING.

COR: $k[x_1, \ldots, x_n]$ IS A NOETHERIAN RING.

$$k[x][y] \cong k[x,y]$$

REMARK: EVERY IDEAL IN $k[x_1,..,x_n]$ IS FINITELY GENERATED.

FOR EVERY $S \subseteq k[x_1,..,x_n]$ $\exists$ $f_1,..,f_n$ SUCH THAT

$$(S) = (f_1,..,f_n).$$

THUS $\boxed{Z(S) = Z((S)) = Z(f_1,..,f_n)}$ ONLY NEED TO CHECK A FINITE # OF POLY'S!

Q: ARE Z AND I INVERSE MAPS?

(GROUPWORK)

CONSIDER $J = (x^2) \subseteq k[x,y]$. DOES $I(Z(J)) = J$?

DEF: LET $Y \subseteq (X, \mathcal{J})$ BE A SUBSET OF A TOPOLOGICAL SPACE. THE CLOSURE OF Y IS DEFINED TO BE

$$\overline{Y} = \bigcap_{\substack{Y \subseteq Z \subseteq X \\ Z \text{ CLOSED}}} Z$$

THE INTERSECTION OF ALL CLOSED SETS CONTAINING Y.

IN PARTICULAR, IF Z IS CLOSED AND $Y \subseteq Z$, THEN $\overline{Y} \subseteq Z$. IE $\overline{Y}$ IS THE SMALLEST CLOSED SET CONTAINING Y.

PROP: LET $A = k[x_1, .., x_n]$.

(a) IF $T_1 \subseteq T_2 \subseteq A$, THEN $Z(T_2) \subseteq Z(T_1)$.

(b) IF $Y_1 \subseteq Y_2 \subseteq \mathbb{A}^n_k$, THEN $I(Y_2) \subseteq I(Y_1)$ ORDER REVERSING

(c) $Y_1, Y_2 \subseteq \mathbb{A}^n_k \Rightarrow I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$

(d) IF $J \subseteq A$ IS AN IDEAL, THEN $\boxed{I(Z(J)) = \sqrt{J}}$, THE RADICAL OF J

$$\sqrt{J} = \{ f \in A \mid f^n \in J \text{ FOR SOME } n \in \mathbb{N} \}$$

(e) IF $Y \subseteq \mathbb{A}^n_k$ IS ANY SUBSET, $\boxed{Z(I(Y)) = \overline{Y}}$.