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A.G. CLASS 5

(ROUGHLY FOLLOWING REID §3 + HARTSHORNE §1)

GROUPWORK: LET $A = k[x_1, \ldots, x_n]$ AS USUAL.

(1) $T_1 \subseteq T_2 \subseteq A \Rightarrow Z(T_2) \subseteq Z(T_1) \subseteq \mathbb{A}^n_k$.

(2) $X_1 \subseteq X_2 \subseteq \mathbb{A}^n_k \Rightarrow I(X_2) \subseteq I(X_1) \subseteq A$

(3) $Z(I_1 \cap I_2) = Z(I_1) \cup Z(I_2)$

(4) $Z\left(\bigcup_{\lambda \in} I_\lambda\right) = \bigcap_{\lambda} Z(I_\lambda)$

(5) $X \subseteq Z(I(X))$ AND $X = Z(I(X)) \iff X$ IS ALGEBRAIC

(6) $I(Z(J)) \supseteq J$ AND INCLUSION MAY BE STRICT.

PROPOSITION:

(a) LET $X \subseteq \mathbb{A}^n_k$ BE ALGEBRAIC. X IS IRREDUCIBLE IFF AND ONLY IF $I(X)$ IS PRIME.

(b) ANY ALGEBRAIC SET HAS A UNIQUE DECOMPOSITION

$$X = X_1 \cup \ldots \cup X_r$$

WHERE X_i IS IRREDUCIBLE AND $X_i \not\subseteq X_j$ FOR $i \neq j$.

THE X_i ARE CALLED THE IRREDUCIBLE COMPONENTS OF X.

Pf: (a) WE PROVE X IS REDUCIBLE $\iff$ $I(X)$ NOT PRIME

SUPPOSE $X = X_1 \cup X_2$ WITH $X_1, X_2 \subsetneq X$ ALGEBRAIC SUBSETS, AND PROPER. SINCE $X_1 \subsetneq X$, $I(X) \subsetneq I(X_1)$. THUS $\exists$ $f_1 \in I(X_1) \setminus I(X)$. SIMILARLY $\exists$ $f_2 \in I(X_2) \setminus I(X)$. LET $p \in X$. THEN $p \in X_1$ OR $p \in X_2$. THUS $f_1(p) = 0$ or $f_2(p) = 0$. THUS $(f_1 f_2)(p) = 0$. HENCE $f_1 f_2 \in I(X)$. HENCE FORCE $I(X)$ IS NOT PRIME.

CONVERSELY, SUPPOSE $I(X)$ IS NOT PRIME. THEN $\exists$ $f_1, f_2 \notin I(X)$ BUT $f_1 f_2 \in I(X)$. LET $I_1 = (I(X), f_1)$, $X_1 = Z(I_1)$. THEN $I(X) \subseteq I_1 \Rightarrow Z(I_1) \subseteq Z(I(X))$, IE $X_1 \subseteq X$. DEFINE X_2 ANALOGOUSLY, AND WE HAVE $X_2 \subseteq X$.

LET $p \in X$. THEN $f_1 f_2(p) = 0$. SO $f_1(p) = 0$ or $f_2(p) = 0$. THEN $p \in X_1$ or $p \in X_2$ (GW: WHY?) (AS $p \in Z(I_1)$ or $p \in Z(I_2)$)

THUS $X = X_1 \cup X_2$.

GROUPWORK: LET R BE A COMMUTATIVE RING W/1. TFAE

(1) R IS NOETHERIAN, IE IT SATISFIES THE ASCENDING CHAIN CONDITION FOR IDEALS

(2) EVERY IDEAL IS FINITELY GENERATED

IE FOR ANY IDEAL I IN R $\exists$ $f_1, .., f_m \in I$ SUCH THAT

$$I = (f_1, .., f_m)$$

(3) EVERY NONEMPTY SET OF IDEALS HAS A MAX'L ELEMENT (ORDERED BY INCLUSION)

HINT (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2)