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ALGEBRAIC GEOMETRY CLASS 8

RECALL: FOR ANY $Y \subseteq \mathbb{A}^n_k$, $Z(I(Y)) = \overline{Y}$

(NOTE: WE'RE USING THE ZARISKI TOPOLOGY ON $\mathbb{A}^n_k$ HERE, AS USUAL.)

(RECALL THE DEFINITION OF $\overline{Y}$)

PF: WE SHOW $\overline{Y} \subseteq Z(I(Y))$ AND $Z(I(Y)) \subseteq \overline{Y}$.

LET $p \in Y$. THEN $\forall f \in (I(Y))$, $f(p) = 0$. THUS $p \in Z(I(Y))$.

HENCE $Y \subseteq Z(I(Y))$. BUT $Z(I(Y))$ IS MANIFESTLY CLOSED IN THE ZARISKI TOPOLOGY ON $\mathbb{A}^n_k$. THEREFORE $\overline{Y} \subseteq Z(I(Y))$, SINCE $\overline{Y}$ IS THE SMALLEST CLOSED SET CONTAINING Y.

CONVERSELY, LET W BE ANY CLOSED SET CONTAINING Y, $Y \subseteq W$. SINCE W IS CLOSED, $\exists$ AN IDEAL $J \subseteq A$ SUCH THAT $W = Z(J)$. THUS $Y \subseteq Z(J)$. THEREFORE $I(Z(J)) \subseteq I(Y)$.

BUT $J \subseteq I(Z(J))$. SO $J \subseteq I(Y)$. THUS

$$Z(I(Y)) \subseteq Z(J) = W.$$

THUS $Z(I(Y)) \subseteq W$ FOR ANY CLOSED W CONTAINING Y. THUS

$Z(I(Y)) \subseteq \overline{Y}$ AND $Z(I(Y)) = \overline{Y}$. QED

DEF: A TOPOLOGICAL SPACE IS NOETHERIAN IF IT SATISFIES THE DESCENDING CHAIN CONDITION FOR ~~IDEALS~~ CLOSED SUBSETS: FOR ANY SEQUENCE OF CLOSED SUBSETS

$$Y_1 \supseteq Y_2 \supseteq \dots$$

THERE IS AN INTEGER r SUCH THAT $Y_r = Y_{r+1} = \dots$.

EX: $\mathbb{A}^n_k$ IS NOETHERIAN. (GROUPWORK)

$Y_1 \supseteq Y_2 \supseteq \dots \Rightarrow I(Y_1) \subseteq I(Y_2) \subseteq \dots$ BUT A IS A NOETH. RING.

$Y_i = Z(I(Y_i))$ b/c Y_i ARE CLOSED.

DEF: IF X IS A TOPOLOGICAL SPACE, THE DIMENSION OF X IS THE SUPREMUM OF ALL $n \in \mathbb{N}_0$ SUCH THAT $\exists$ CHAIN

$$Z_0 \subsetneq Z_1 \subsetneq \dots \subsetneq Z_n$$

OF DISTINCT IRREDUCIBLE CLOSED SUBSETS OF X. WE DEFINE THE DIMENSION OF AN ALG. SET TO BE ITS TOP. DIM.

EX: WHAT IS THE DIMENSION OF $\mathbb{A}^1$? (GROUPWORK: TRY ZARISKI + METRIC TOPOLOGIES)

DEF: IN A RING A, THE HEIGHT OF A PRIME IDEAL P IS THE SUPREMUM OF ALL $n \in \mathbb{Z}$ SUCH THAT $\exists$ CHAIN

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P$$

OF DISTINCT PRIME IDEALS. THE KRULL DIMENSION OF A IS THE SUP OF ALL HEIGHTS OF PRIME IDEALS OF A.

~~PROP:~~ DEF: FOR AN ALGEBRAIC SUBSET $Y \subseteq \mathbb{A}^n_k$, THE AFFINE COORDINATE RING OF Y IS

$$A(Y) = A/I(Y) = k[x_1, , x_n]/I(Y)$$

EX:

$Y = Z(xy-1)$. $A(Y)$

$X = Z(y-x^2)$ $A(X)$ (GROUPWORK)

$W = \mathbb{A}^1_k$

PROP: FOR ALGEBRAIC $Y \subseteq \mathbb{A}^n_k$

$$\dim Y = \text{Krull } \dim A(Y).$$