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ALGEBRAIC GEOMETRY CLASS 9

RECALL:

$(+,\times)$ RING ALGEBRA.

GROUP MODULE

$+$ $+, \cdot$

DEF: ~~A MODU~~ LET R BE A COMMUTATIVE RING W/ 1. A (LEFT) R MODULE IS AN ABELIAN GROUP M TOGETHER WITH AN ACTION OF R ON M

$$R \times M \to M$$

$$(r, x) \mapsto r \cdot x = rx$$

SUCH THAT

(1) $\forall\, r, s \in R$, $r(sx) = (rs)x$

(2) $r(x+y) = rx + ry$

$(r+s)x = rx + sx$

(3) $1x = x$.

EX: IF $R = k$ A FIELD, M IS A k-VECTOR SPACE.

DEF: A k-ALGEBRA IS A RING A WHICH IS ALSO A k-MODULE.

A IS FINITELY GENERATED IF $\exists\, a_1, \dots, a_n \in A$ SUCH THAT EVERY ELEMENT OF A CAN BE EXPRESSED AS A POLYNOMIAL IN $a_1, \dots, a_n$ WITH COEFFICIENTS IN k. WE THEN DENOTE

$$A = k[a_1, \dots, a_n].$$

NOTE: THERE MAY BE RELATIONS AMONG THE a_i

THM: EVERY F.G. k-ALG IS ISOMORPHIC TO

$$k[x_1, \dots, x_n]/I$$

FOR SOME IDEAL $I \subseteq k[x_1, \dots, x_n]$

THM (HARD FACT):

LET $|k| = \infty$, $A = k[a_1, \dots, a_n]$ F.G. k-ALG.

IF A IS A FIELD, THEN A IS ALGEBRAIC OVER k

NOTE: ALG OVER k $\Rightarrow$ EACH ELT OF A IS ZERO OF A POLY. W/ COEFF. IN k.

NULLSTELLENSATZ $\quad k = \bar{k}$.

(1) $A = k[x_1,..,x_n]$. EVERY MAX'L IDEAL IS OF FORM

$$m_p = (x_1 - a_1, .., x_n - a_n) \quad \text{SOME} \quad p = (a_1, .., a_n)$$

(2) $J \subseteq A$ IDEAL. $J \neq (1) \Rightarrow Z(J) \neq \emptyset$

(3) $I(Z(J)) = \sqrt{J}$

PF:

(1) LET $m \subseteq A$ BE MAX'L. LET $K = A/m$.

$$\begin{tikzcd} k \arrow[r, hook, "\iota"] \arrow[rr, bend right, "\varphi"'] & k[x_1,..,x_n] = A \arrow[r, two heads, "\pi"] & A/m = K \end{tikzcd}$$

NOTE K IS A FIELD B/C m MAX'L, F.G. BY $\{\overset{b_i}{\overset{\shortparallel}{[x_i + m]}}\}$.

BY HARD FACT, $\varphi : k \to K$ IS AN ALGEBRAIC EXTENSION.

BUT $k = \bar{k}$. THUS φ IS AN ISOMORPHISM.

NOW LET $b_i = \pi(x_i) = [x_i + m]$, AND LET $a_i = \varphi^{-1}(b_i)$.

THEN $x_i - a_i \in \ker \pi = m$. THUS $(a_1 - x_1, .., x_n - a_n) \subseteq m$.

BUT $\underline{(x_1 - a_1, .., x_n - a_n) \text{ IS MAX'L}}$. THUS $m = (x_1 - a_1, .., x_n - a_n)$.

(WHY?)

(1) ⇒ (2) SUPPOSE $J \neq (1) = A$. BY A.C.C. $\exists$ MAX'L IDEAL m

s.t. $J \subseteq m$. BUT $m = (x_1 - a_1, \dots, x_n - a_n)$ FOR SOME $a_1, \dots, a_n \in A$ BY (1)

THEN $f(p) = 0 \ \forall f \in J$, WHERE $p = (a_1, \dots, a_n)$. THUS $p \in Z(J)$, $Z(J) \neq \emptyset$.

(2) ⇒ (3) LET $J \subseteq A$, $f \in A$. DEFINE (OUT OF THE BLUE)

$$J' = (J, fy - 1) \subseteq k[x_1, \dots, x_n, y] = A[y].$$

NOTE: FOR $q = (a_1, \dots, a_n, b) \in \mathbb{A}^{n+1}_k$, $q \in Z(J') \Rightarrow$

$$g(a_1, \dots, a_n) = 0 \ \forall g \in J \quad (\text{IE } p \in Z(J))$$

AND $f(p) \cdot b = 1$, IE $f(p) \neq 0$, $b = 1/f(p)$

SO J KEEPS TRACK OF PTS WHERE F DOESN'T VANISH!

SUPPOSE $f \in I(Z(J))$ IE $f(p) = 0 \ \forall p \in Z(J)$. THEN

$Z(J') = \emptyset$. THUS, BY (2), $1 \in J'$ AND

$$1 = \sum_{i=1}^{m} g_i f_i + g_0 (fy - 1) \in A[y] = k[x_1, \dots, x_n, y] \quad (\star)$$

WITH $f_i \in J$, $g_i, g_0 \in k[x_1, \dots, x_n, y]$.

SUPPOSE N IS THE HIGHEST POWER OF y IN g_i, g_0.

LET $G_i = f^N g_i$, AND $\star$ BECOMES

$G_i(x_1, \dots, x_n, fy) =$.

$$f^N = \sum' G_i(x_1,.,x_n, fy) f_i + G_0(x_1,.,x_n, fy)(fy-1)$$

REDUCE MOD $(fy-1)$

$$f^N = \sum' h_i(x_1, x_n) f_i \in k[x_1, x_n]/(fy-1)$$

BUT $k[x_1, x_n] \to k[x_1, x_n][y]/(fy-1)$ IS INJECTIVE!

$k[x_1,..x_n] \hookrightarrow k[x_1, x_n][f^{-1}]$

THUS $f^N = \sum h_i(x_1,, x_n) f_i \in k[x_1, x_n]$

IE $f^N \in J$. QED.