Knot Theory and Khovanov Homology

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Knot Theory

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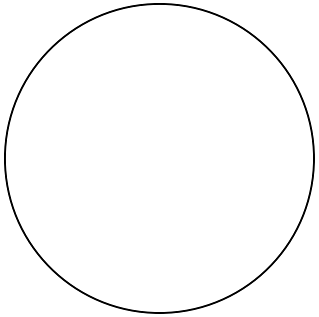

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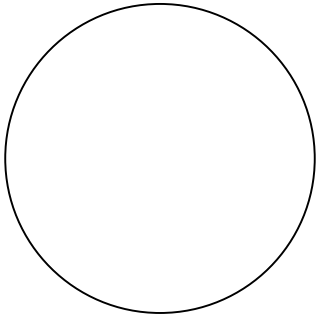

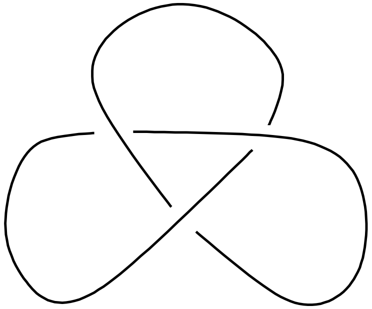

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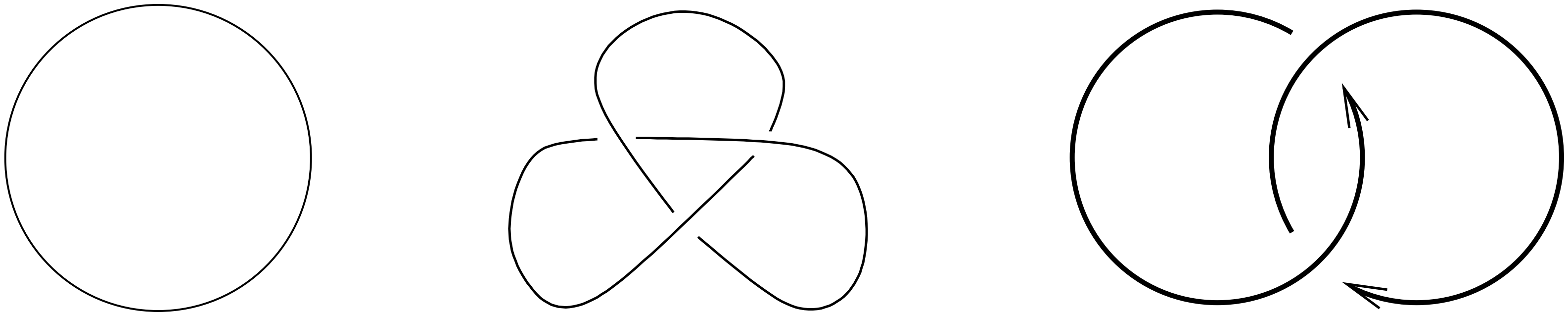

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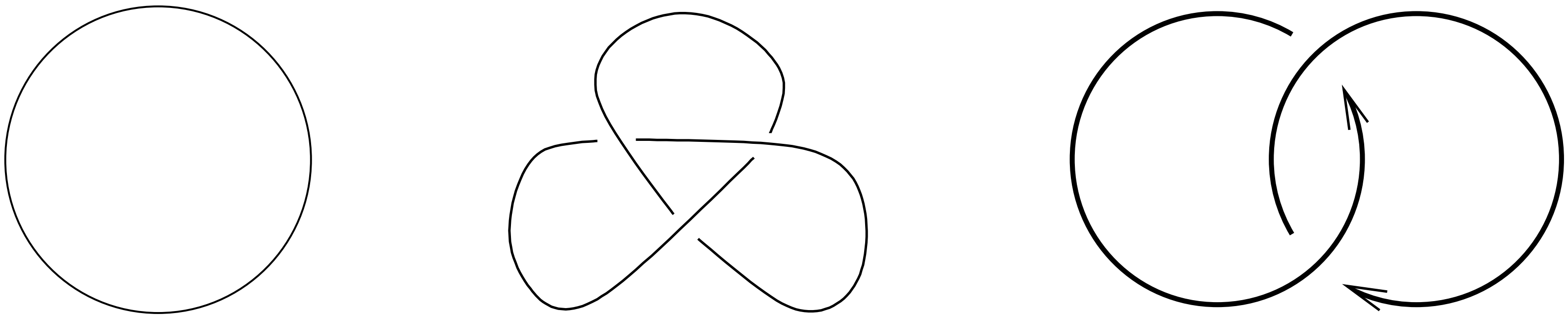

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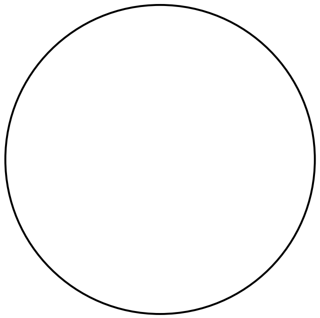

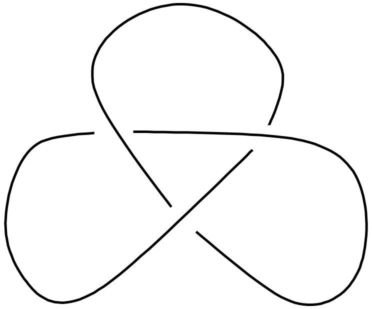

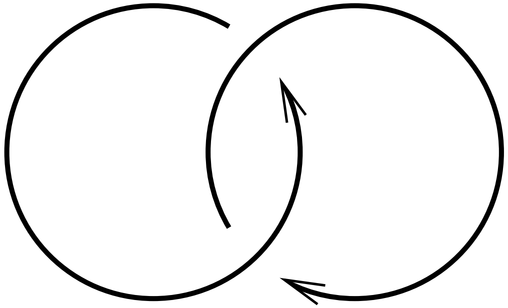

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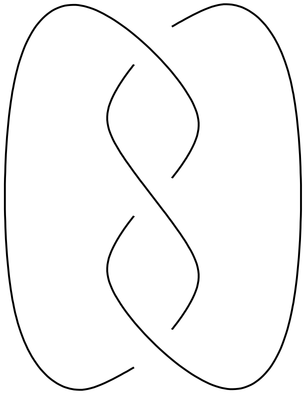

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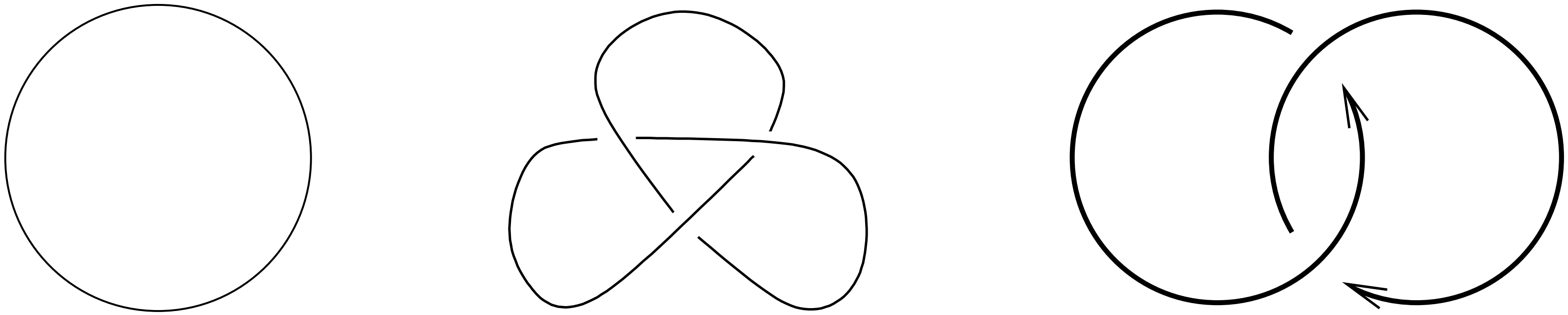

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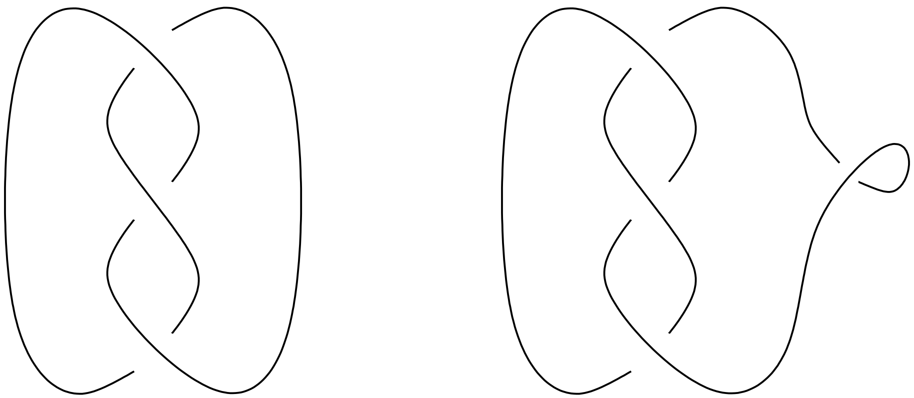

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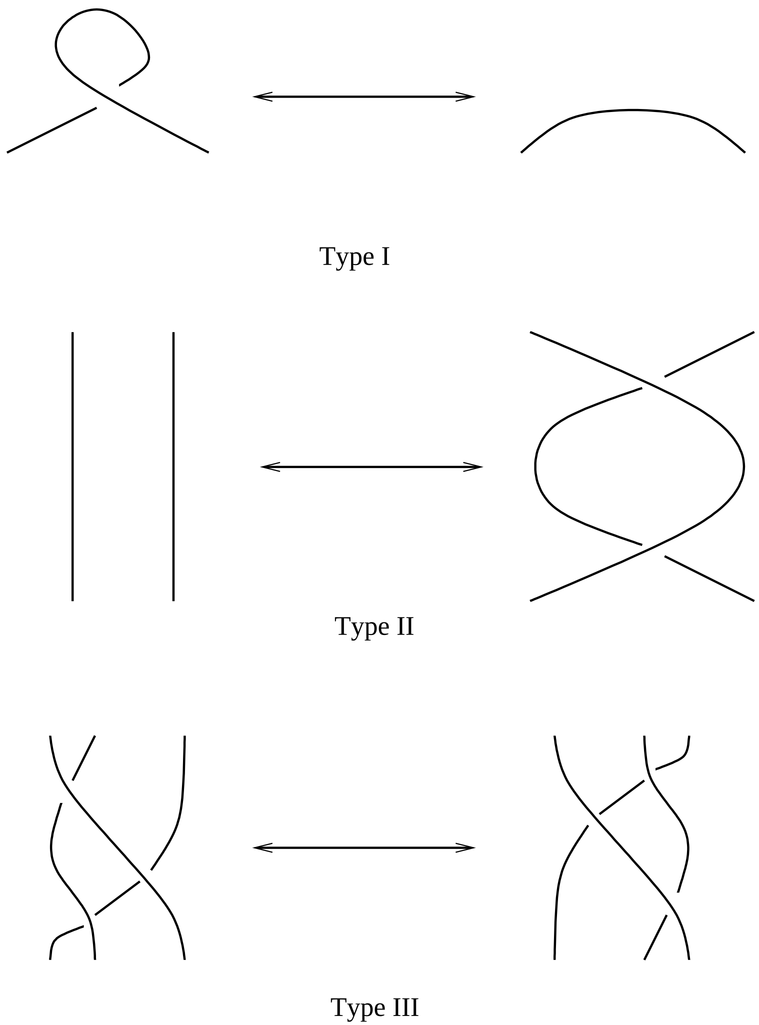

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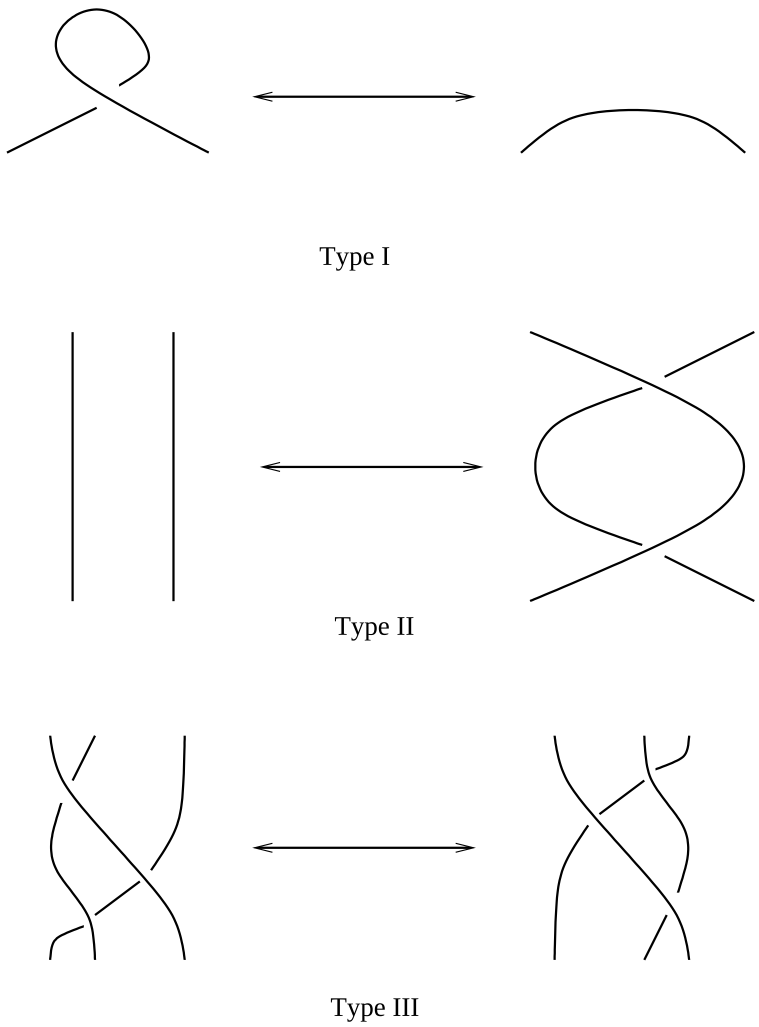
- Link Invariant: Property unchanged by these moves



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  - Kauffman's Bracket

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$$
\begin{aligned}
<\emptyset>= &\quad 1 \\
<\text{unknot} \sqcup L>= &\quad (q+q^{-1}) < L > \\
< D >= &\quad < D_+ > -q < D_- >
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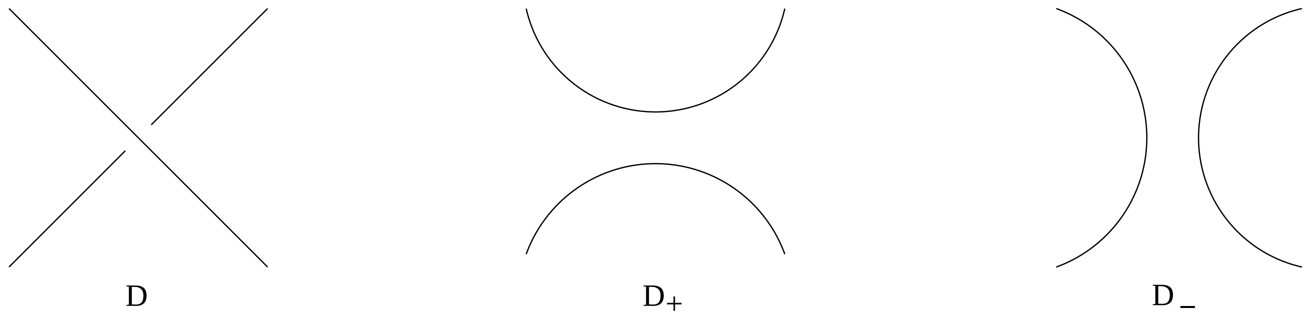
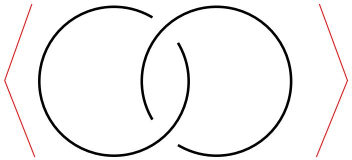


Kauffman's Bracket for the Hopf Link



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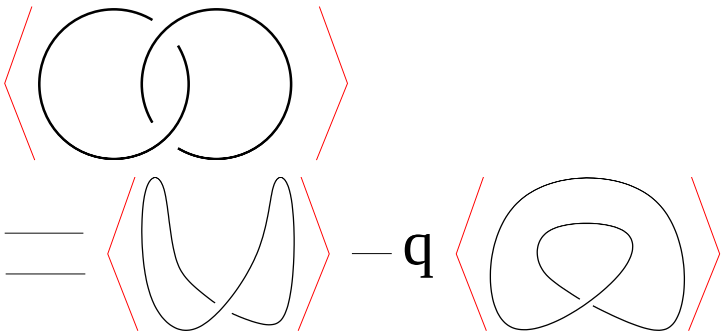

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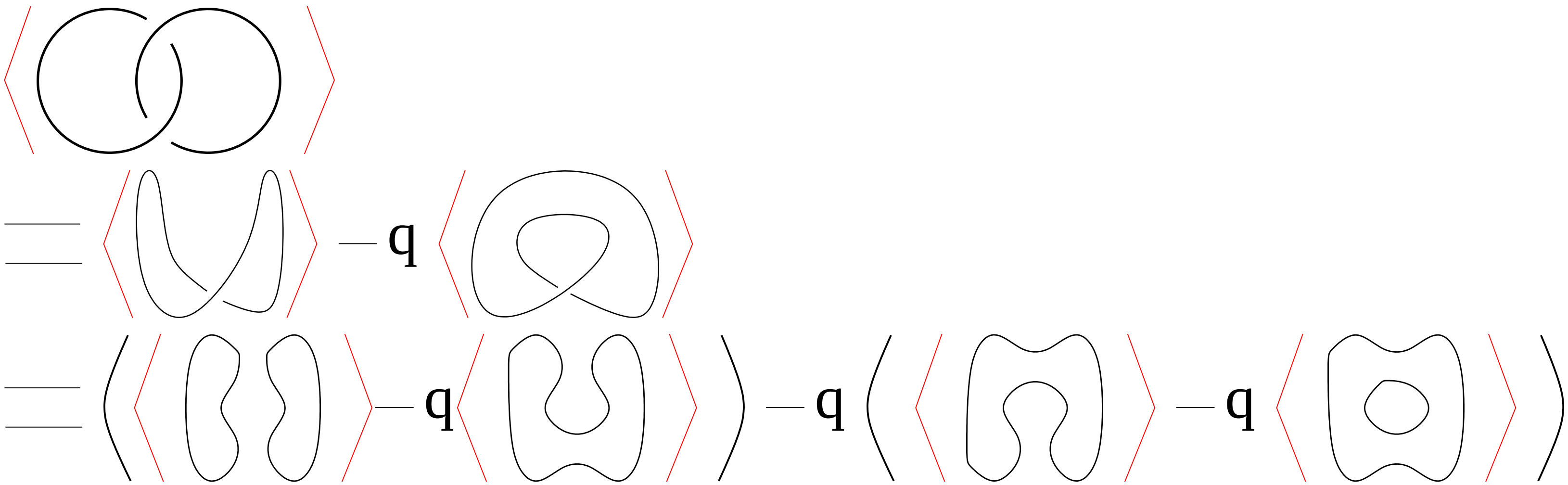

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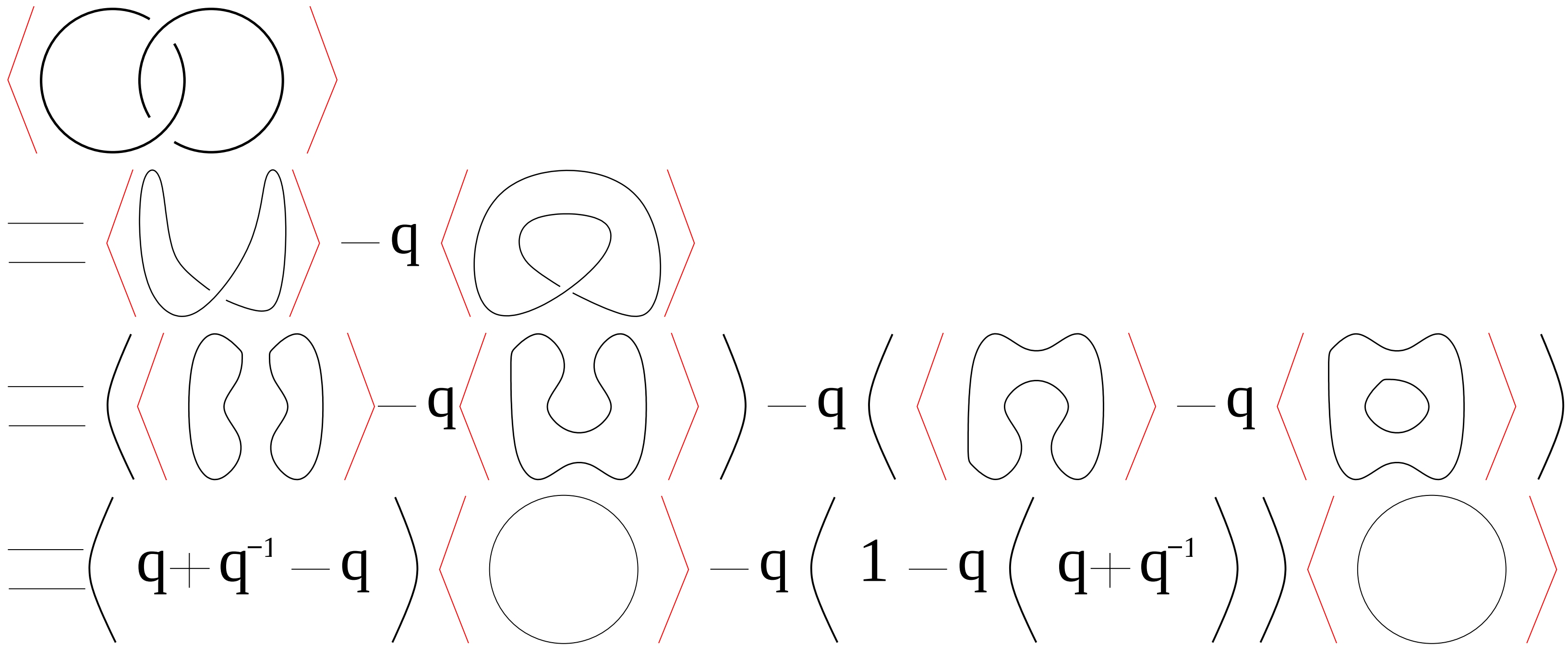

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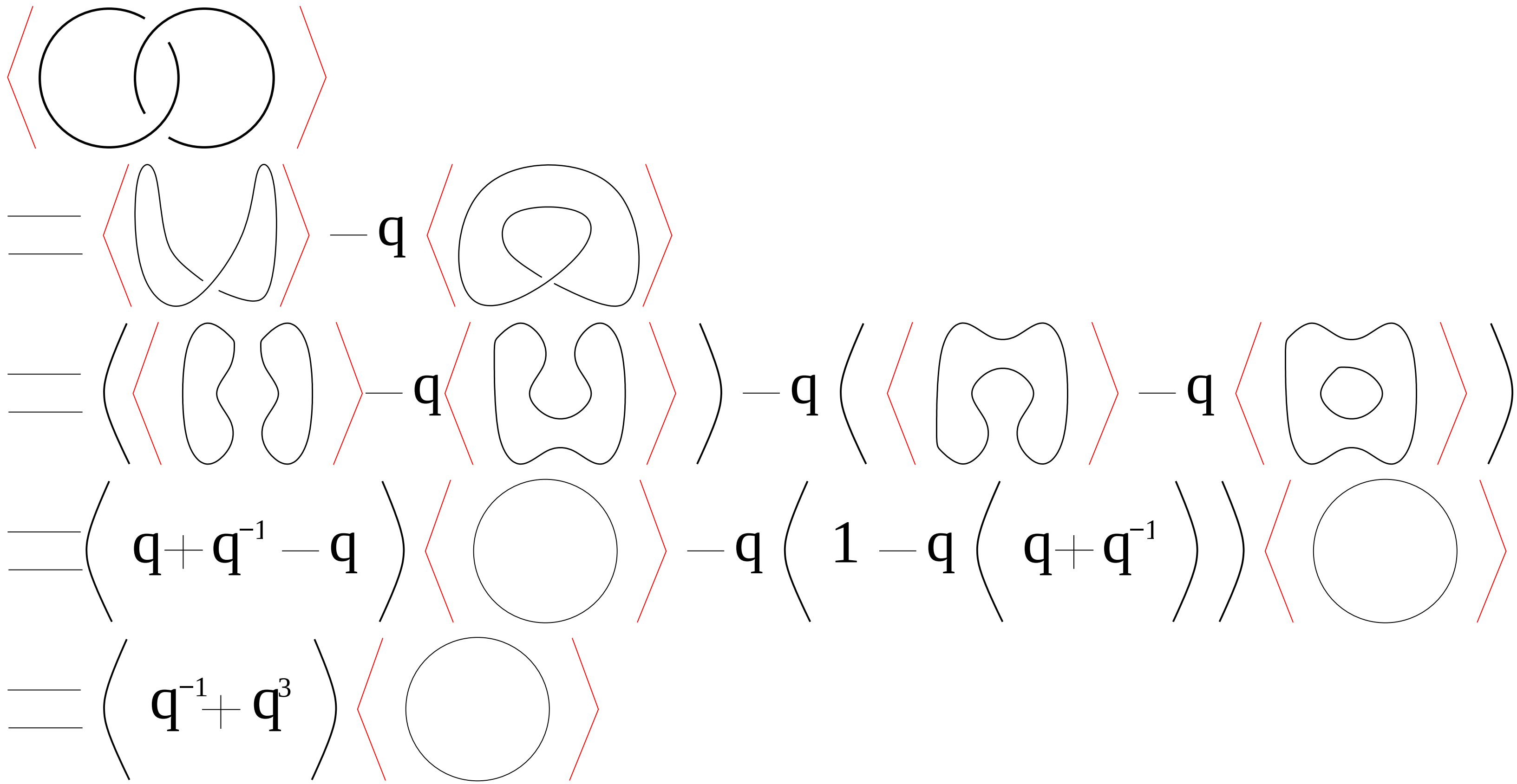

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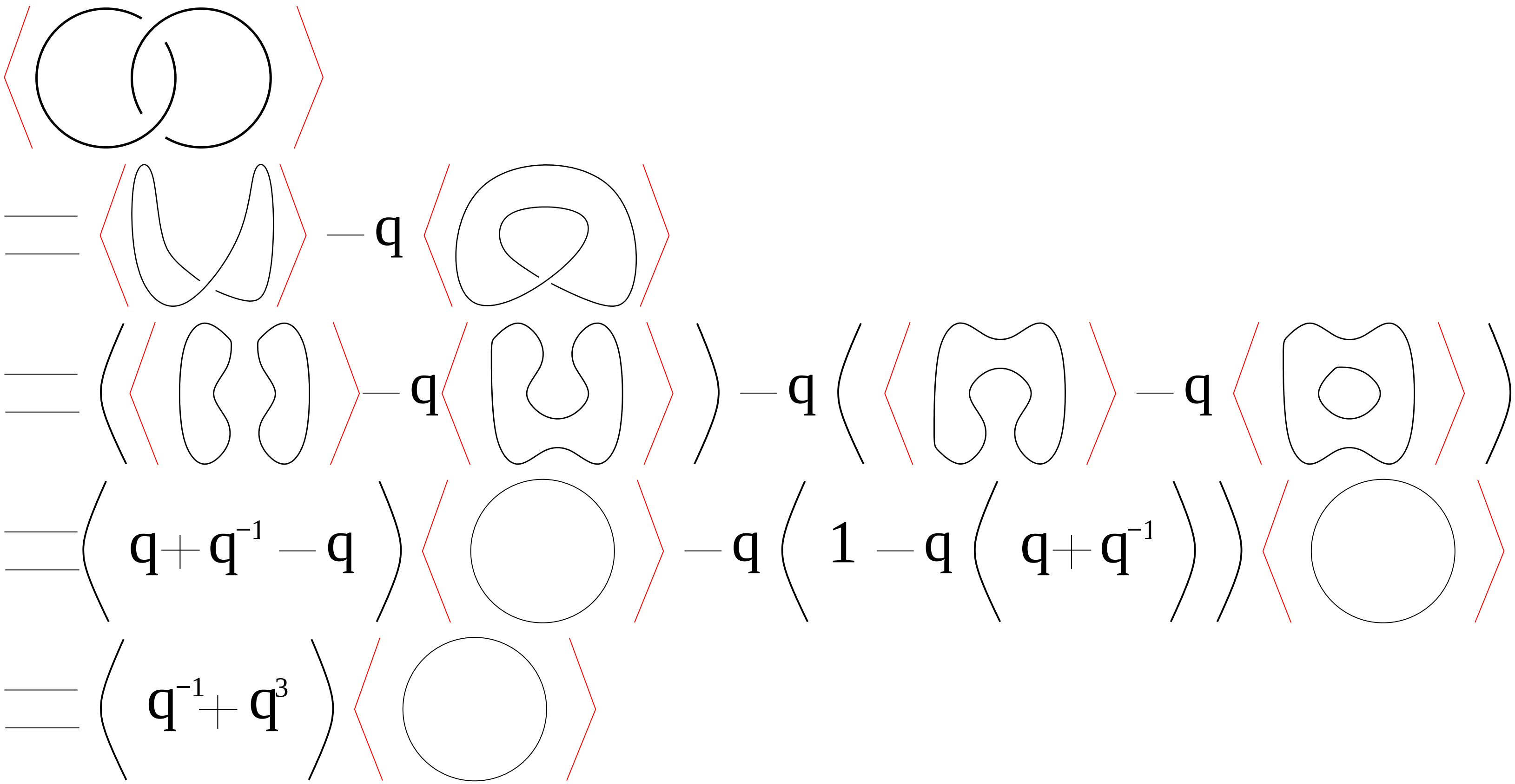

$$= \left(q^{-1} + q^{3} \right) \left(q + q^{-1} \right)$$

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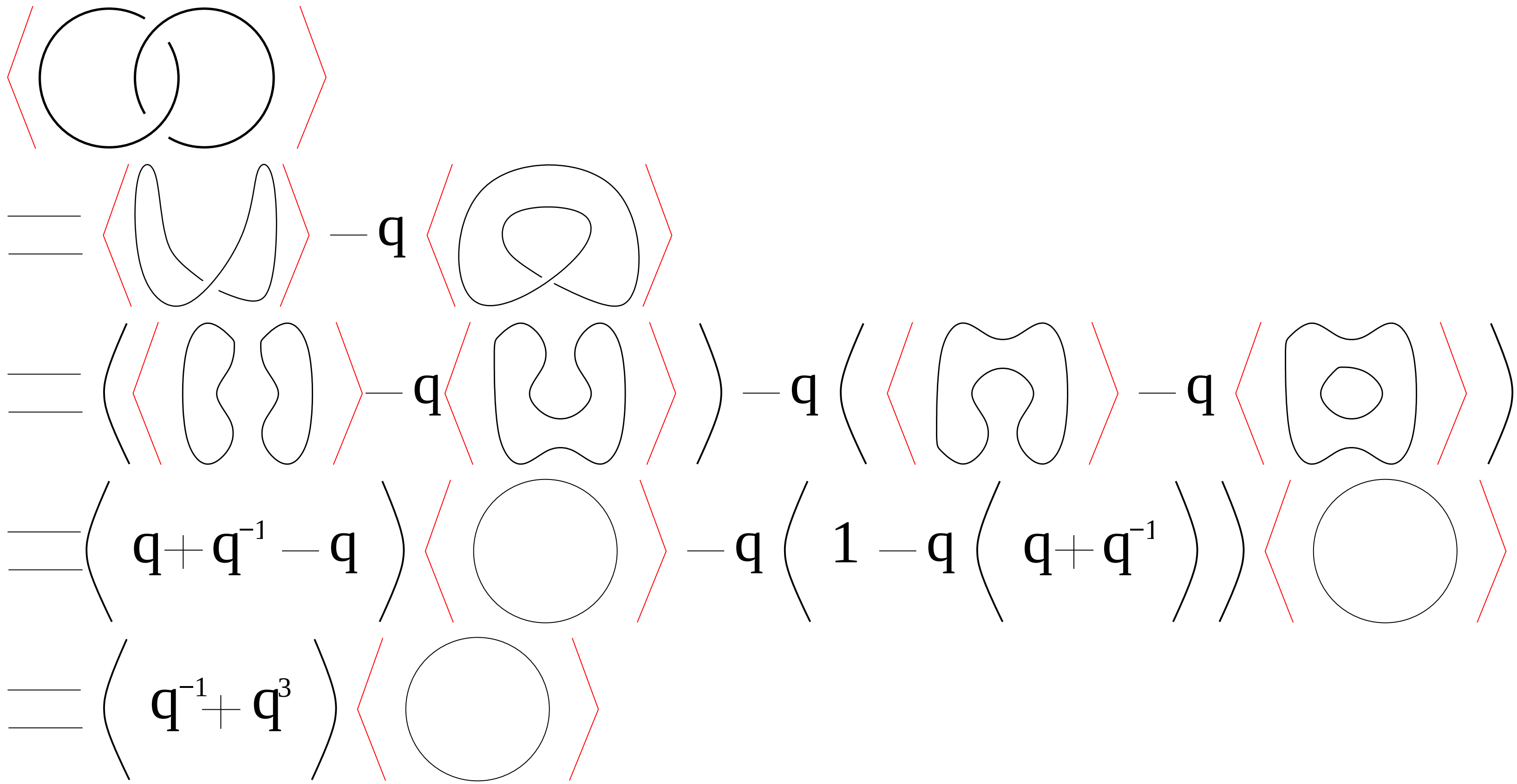

$$= \left(q^{-1} + q^{3} \right) \left(q + q^{-1} \right)$$

$$= q^{-2} + 1 + q^{2} + q^{4}$$

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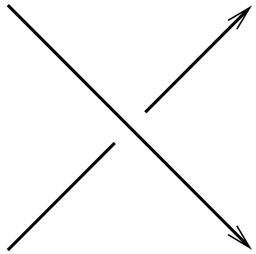

positive

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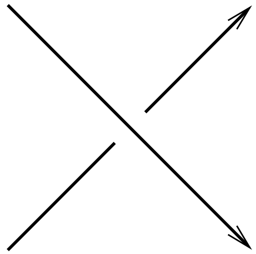

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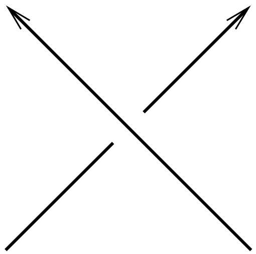

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Let n_+ be the number of positive crossings and n_- be the number of negative crossings.

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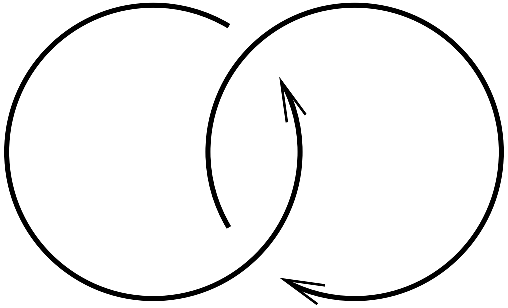
positive negative

Let n_+ be the number of positive crossings and n_- be the number of negative crossings. Then the Jones Polynomial $J_L(q)$ of the link L is:

$$J_L(q) = -1^{n_-} q^{n_+ - 2n_-} < L >$$

Jones Polynomial for the Hopf Link



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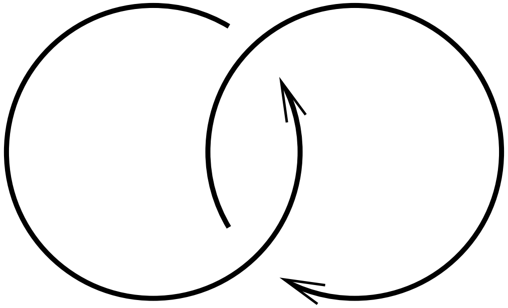

In this diagram $n_+ = 2$ and $n_- = 0$. Then the Jones Polynomial for the Hopf Link is:

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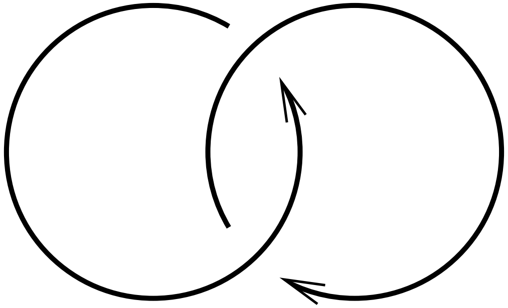

In this diagram $n_+ = 2$ and $n_- = 0$. Then the Jones Polynomial for the Hopf Link is:

$$\begin{aligned} J_{hopf}(q) &= (-1)^0 q^2 (q^{-2} + 1 + q^2 + q^4) \\ &= 1 + q^2 + q^4 + q^6 \end{aligned}$$

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- Categorification of the Jones polynomial
- Uses the algebra $A = < 1, x >$ over a field $\mathbb{F}$ with the following operations:

$$A \otimes A \xrightarrow{m} A, m(1 \otimes b) = m(b \otimes 1) = b, \quad m(x \otimes x) = 0$$

$$A \xrightarrow{\Delta} A \otimes A, \Delta(1) = 1 \otimes x + x \otimes 1, \quad \Delta(x) = x \otimes x$$

Khovanov Complex

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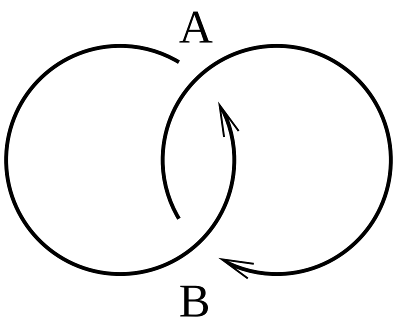

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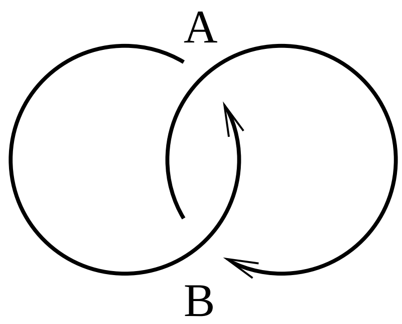

- Two ways of smoothing a crossing:

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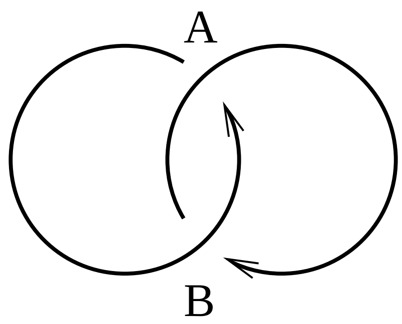

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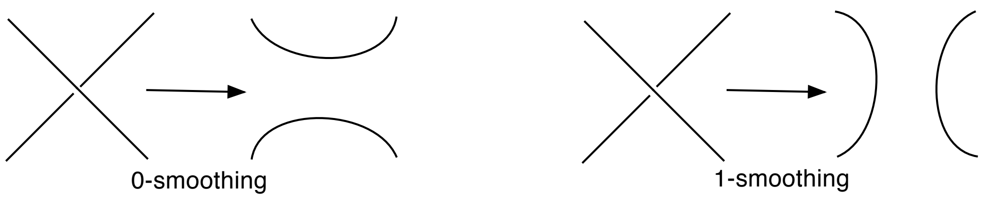

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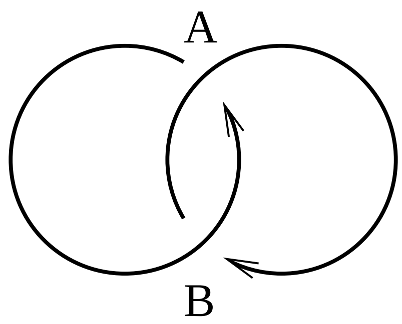

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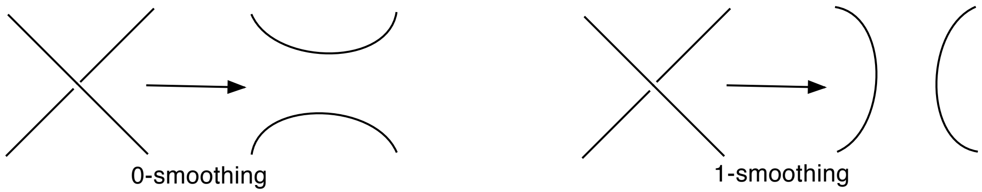

- For each vertex α of the n-dimensional cube it assigns a particular way of smoothing the knot: $\alpha \to$ state α

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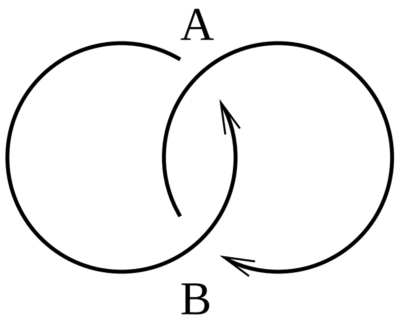

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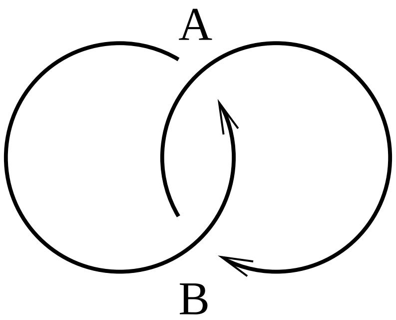

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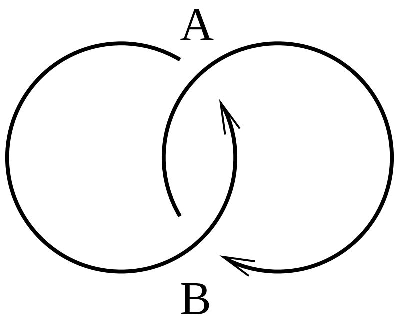

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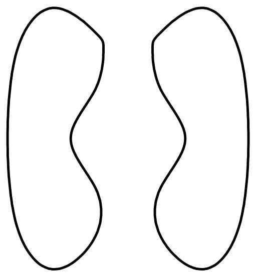

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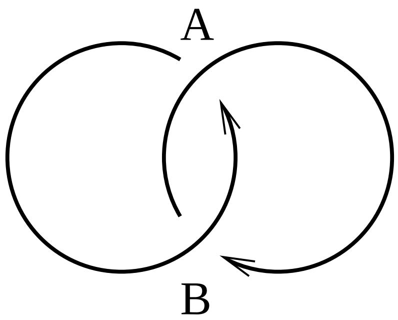

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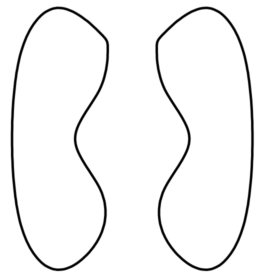

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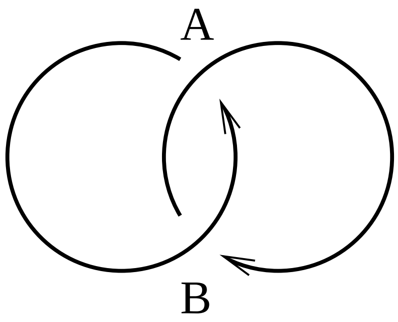

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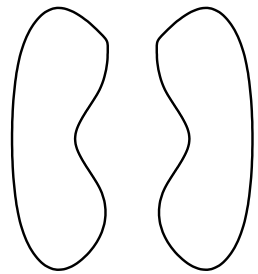

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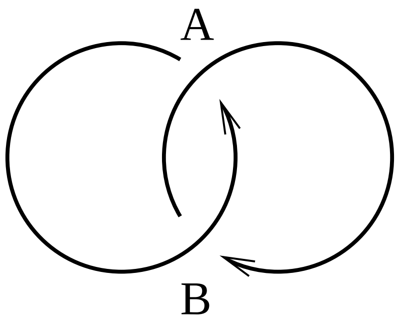

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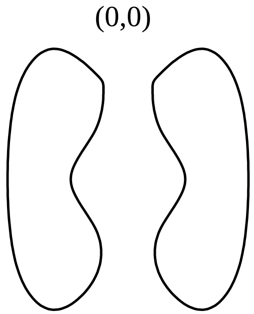

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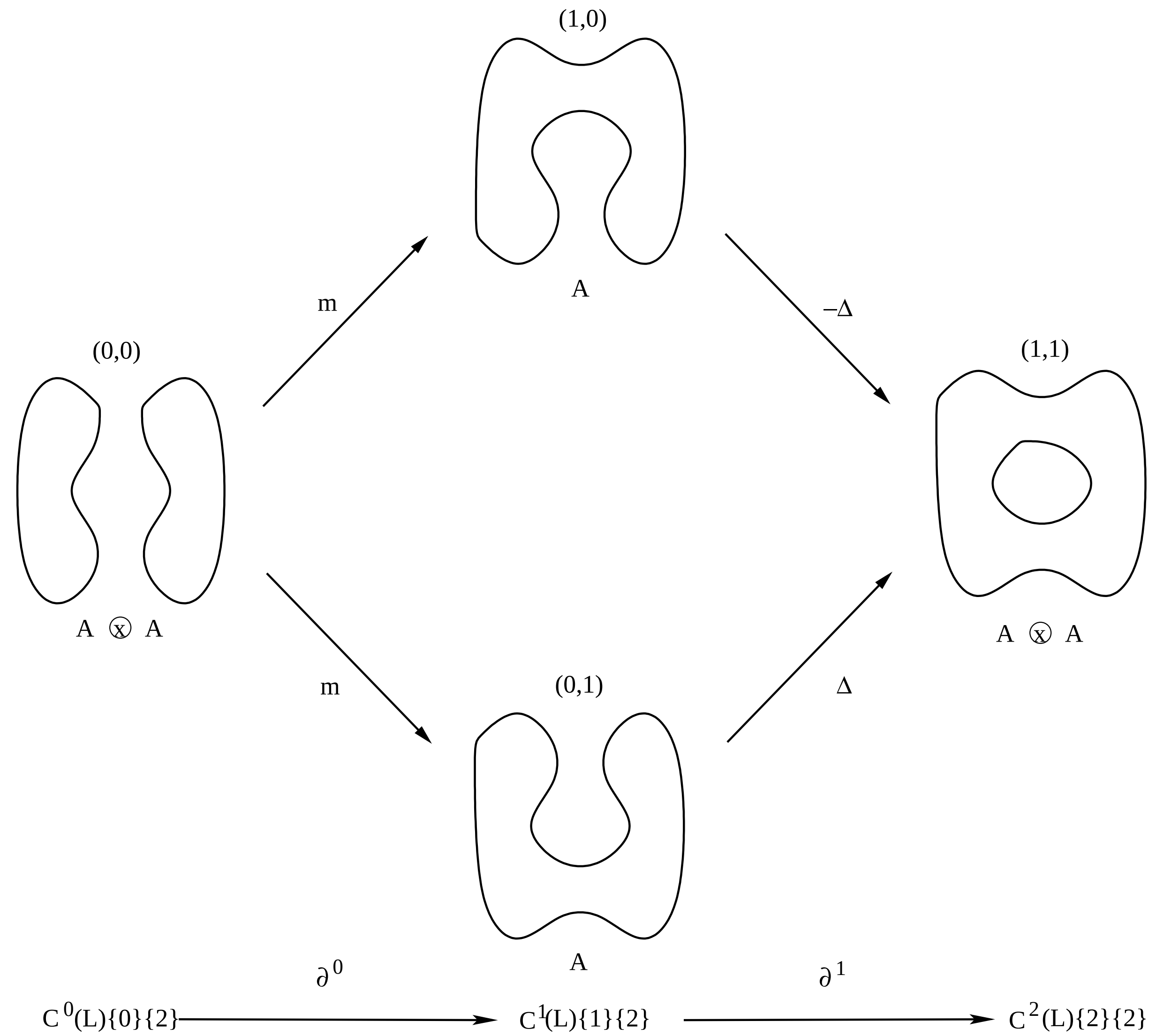
- Let $C_i = \bigoplus_{|\alpha|=i} [L]_\alpha$ to get the complex with ∂-operator defined by the operations on the algebra A

Complex for the Hopf link



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The homology $H^i(L)$ of the complex $C(L)$ is an invariant for the link L and it is called the Khovanov homology of L. From this groups one can obtain the Jones polynomial. In general, the Khovanov homology is stronger (invariant) than the Jones polynomial.

KH for the Hopf Link

The complex: $0 \longrightarrow C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \longrightarrow 0$

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-
  - $(1,0) \xrightarrow{\partial^1} -1 \otimes x - x \otimes$
  - $(x,0) \xrightarrow{\partial^1} -x \otimes x$
  - $(0,1) \xrightarrow{\partial^1} 1 \otimes x + x \otimes 1$
  - $(0,x) \xrightarrow{\partial^1} x \otimes x$

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  - $1 \otimes x \xrightarrow{\partial^0} (x,x)$
  - $x \otimes x \xrightarrow{\partial^0} (0,0)$
- $Z^0 =< 1 \otimes x - x \otimes 1, x \otimes x >$

- - $(1,0) \xrightarrow{\partial^1} -1 \otimes x - x \otimes$
  - $(x,0) \xrightarrow{\partial^1} -x \otimes x$
  - $(0,1) \xrightarrow{\partial^1} 1 \otimes x + x \otimes 1$
  - $(0,x) \xrightarrow{\partial^1} x \otimes x$

KH for the Hopf Link

The complex: $0 \longrightarrow C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \longrightarrow 0$

-
  - $1 \otimes 1 \xrightarrow{\partial^0} (1,1)$
  - $x \otimes 1 \xrightarrow{\partial^0} (x,x)$
  - $1 \otimes x \xrightarrow{\partial^0} (x,x)$
  - $x \otimes x \xrightarrow{\partial^0} (0,0)$
- $Z^0 =< 1 \otimes x - x \otimes 1, x \otimes x >$
- $B^0 =< (1,1), (x,x) >$
-
  - $(1,0) \xrightarrow{\partial^1} -1 \otimes x - x \otimes$
  - $(x,0) \xrightarrow{\partial^1} -x \otimes x$
  - $(0,1) \xrightarrow{\partial^1} 1 \otimes x + x \otimes 1$
  - $(0,x) \xrightarrow{\partial^1} x \otimes x$

KH for the Hopf Link

The complex: $0 \to C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \to 0$

-
  - $1 \otimes 1 \xrightarrow{\partial^0} (1,1)$
  - $x \otimes 1 \xrightarrow{\partial^0} (x,x)$
  - $1 \otimes x \xrightarrow{\partial^0} (x,x)$
  - $x \otimes x \xrightarrow{\partial^0} (0,0)$
-
  - $(1,0) \xrightarrow{\partial^1} -1 \otimes x - x \otimes$
  - $(x,0) \xrightarrow{\partial^1} -x \otimes x$
  - $(0,1) \xrightarrow{\partial^1} 1 \otimes x + x \otimes 1$
  - $(0,x) \xrightarrow{\partial^1} x \otimes x$
- $Z^0 =< 1 \otimes x - x \otimes 1, x \otimes x >$
- $Z^1 =< (1,1), (x,x) >$
- $B^0 =< (1,1), (x,x) >$

KH for the Hopf Link

The complex: $0 \to C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \to 0$

-
  - $1 \otimes 1 \xrightarrow{\partial^0} (1,1)$
  - $x \otimes 1 \xrightarrow{\partial^0} (x,x)$
  - $1 \otimes x \xrightarrow{\partial^0} (x,x)$
  - $x \otimes x \xrightarrow{\partial^0} (0,0)$
- $Z^0 =< 1 \otimes x - x \otimes 1, x \otimes x >$
- $B^0 =< (1,1), (x,x) >$
-
  - $(1,0) \xrightarrow{\partial^1} -1 \otimes x - x \otimes$
  - $(x,0) \xrightarrow{\partial^1} -x \otimes x$
  - $(0,1) \xrightarrow{\partial^1} 1 \otimes x + x \otimes 1$
  - $(0,x) \xrightarrow{\partial^1} x \otimes x$
- $Z^1 =< (1,1), (x,x) >$
- $B^1 =< 1 \otimes x + x \otimes 1, x \otimes x$

KH for the Hopf Link

The complex: $0 \to C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \to 0$

- - $1 \otimes 1 \xrightarrow{\partial^0} (1,1)$
  - $x \otimes 1 \xrightarrow{\partial^0} (x,x)$
  - $1 \otimes x \xrightarrow{\partial^0} (x,x)$
  - $x \otimes x \xrightarrow{\partial^0} (0,0)$
- - $(1,0) \xrightarrow{\partial^1} -1 \otimes x - x \otimes$
  - $(x,0) \xrightarrow{\partial^1} -x \otimes x$
  - $(0,1) \xrightarrow{\partial^1} 1 \otimes x + x \otimes 1$
  - $(0,x) \xrightarrow{\partial^1} x \otimes x$
- $Z^0 =< 1 \otimes x - x \otimes 1, x \otimes x >$
- $Z^1 =< (1,1), (x,x) >$
- $B^0 =< (1,1), (x,x) >$
- $B^1 =< 1 \otimes x + x \otimes 1, x \otimes x$

$H^0 \;=< \; x \otimes 1 - 1 \otimes x, x \otimes x \;>$

KH for the Hopf Link

The complex: $0 \rightarrow C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \rightarrow 0$

- - $1 \otimes 1 \xrightarrow{\partial^0} (1,1)$
  - $x \otimes 1 \xrightarrow{\partial^0} (x,x)$
  - $1 \otimes x \xrightarrow{\partial^0} (x,x)$
  - $x \otimes x \xrightarrow{\partial^0} (0,0)$
- - $(1,0) \xrightarrow{\partial^1} -1 \otimes x - x \otimes$
  - $(x,0) \xrightarrow{\partial^1} -x \otimes x$
  - $(0,1) \xrightarrow{\partial^1} 1 \otimes x + x \otimes 1$
  - $(0,x) \xrightarrow{\partial^1} x \otimes x$
- $Z^0 =< 1 \otimes x - x \otimes 1, x \otimes x >$
- $Z^1 =< (1,1), (x,x) >$
- $B^0 =< (1,1), (x,x) >$
- $B^1 =< 1 \otimes x + x \otimes 1, x \otimes x$

$H^0 =< x \otimes 1 - 1 \otimes x, x \otimes x >$, $H^1 = 0$

KH for the Hopf Link

The complex: $0 \longrightarrow C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \longrightarrow 0$

-
  - $1 \otimes 1 \xrightarrow{\partial^0} (1,1)$
  - $x \otimes 1 \xrightarrow{\partial^0} (x,x)$
  - $1 \otimes x \xrightarrow{\partial^0} (x,x)$
  - $x \otimes x \xrightarrow{\partial^0} (0,0)$
-
  - $(1,0) \xrightarrow{\partial^1} -1 \otimes x - x \otimes$
  - $(x,0) \xrightarrow{\partial^1} -x \otimes x$
  - $(0,1) \xrightarrow{\partial^1} 1 \otimes x + x \otimes 1$
  - $(0,x) \xrightarrow{\partial^1} x \otimes x$
- $Z^0 =< 1 \otimes x - x \otimes 1, x \otimes x >$
- $Z^1 =< (1,1), (x,x) >$
- $B^0 =< (1,1), (x,x) >$
- $B^1 =< 1 \otimes x + x \otimes 1, x \otimes x$

$H^0 =< x \otimes 1 - 1 \otimes x, x \otimes x >$, $H^1 = 0$, and

$H^2 =< 1 \otimes 1, 1 \otimes x - x \otimes 1 >$

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