



Oh, the Places You'll Go!

...a journey from Physics to Combinatorics

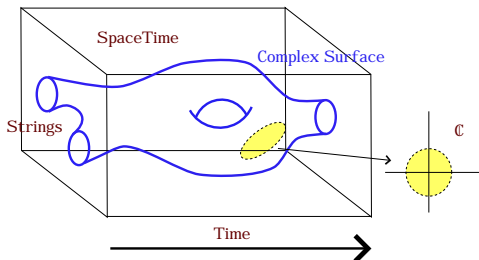
Renzo Cavalieri

Colorado State University

SACNAS 2009

Dr. Seuss

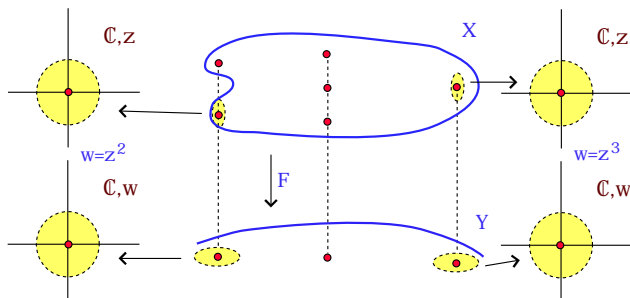
Physicists tell us that matter consists of tiny little **strings** that wiggle their way through Space Time. In doing so they trace **surfaces**, which represent the evolution of a physical system. These surfaces come with a **complex structure**.



It seems reasonable that physicists want to know as much as possible about **analytic functions** between surfaces.

Geometry

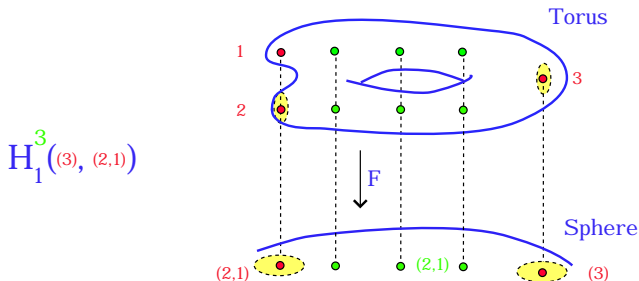
Geometers have been studying such maps for over a century, and know they are **ramified covers**: away from a finite number of points (**branch points**), there are exactly d preimages.



Above a branch point, the local expression of the function is $z \mapsto z^n$, and the collection of the n 's above a branch is called the **ramification profile**.

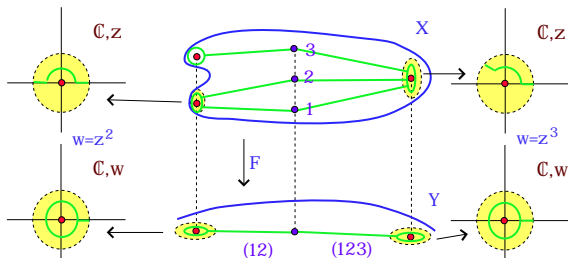
The Quest

QUESTION: how many covers of degree d from a genus g surface to a sphere, with specified ramification profile over two points and generic ramification over r other points?



Such number is called a Hurwitz number, denoted $H_g^r(\alpha, \beta)$.

Algebra gives us a way to compute Hurwitz number via the following construction:



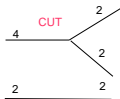
In order to count covers, we instead count $(r + 2)$ -tuples $\sigma_0, \tau_1, \dots, \tau_r, \sigma_\infty$ of permutations of d points such that:

- 1 $(\sigma_0, \sigma_\infty)$ have cycle type (α, β) ;
- 2 τ 's are simple transpositions;
- 3 $\sigma_0 \tau_1 \dots \tau_r \sigma_\infty = Id$
- 4 $\langle \sigma_0, \tau_1, \dots, \tau_r, \sigma_\infty \rangle$ acts transitively on the d points

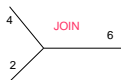
Combinatorics

The **cut and join** equations tell us how a permutation can change when you compose it with a simple transposition:

$$\sigma = (1234)(56) \quad \tau_1 = (13) \quad \Rightarrow \quad \tau_1 \sigma = (12)(34)(56)$$

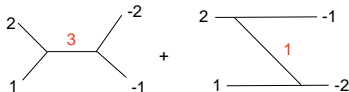


$$\sigma = (1234)(56) \quad \tau_1 = (35) \quad \Rightarrow \quad \tau_1 \sigma = (125634)$$



This gives us a way to compute Hurwitz number by instead counting **weighted trivalent graphs**.

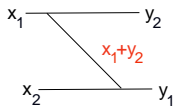
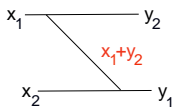
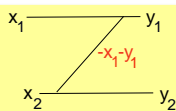
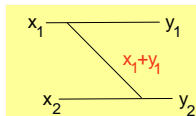
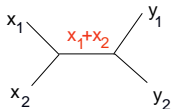
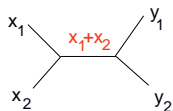
$$H_0^2((2, 1), (2, 1)) = 4 =$$



Sketch of further coolness...

$$x_1 + y_2 > 0$$

$$x_1 + y_1 > 0 \quad x_1 + y_1 = 0 \quad x_1 + y_1 < 0$$



$$2x_1$$

$$2(x_1 + y_1)$$

$$-2y_1$$