Finite Difference
vs.
Finite Volume Method

$$q_t + qq_x = 0$$

Victor Camacho

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Introduction

- Inviscid Burgers Equation
- Similar to standing wave
  - Speed varies by height
  - No diffusion term
  - Shock may develop in finite time
- This PDE has non-hyperbolic solutions

$$q_t + qq_x = 0$$

$$q(-L,t) = q(L,t) = 0$$

$$q(x,0) = \varphi(x)$$

Basic Finite Difference Method

- Conservative form

$$q_t + \left(\frac{q^2}{2} \right)_x = 0$$

- Basic Finite difference method

$$\frac{Q_i^{n+1} - Q_i^{n-1}}{2\Delta t} + \frac{(Q_{i+1}^n)^2 - (Q_{i-1}^n)^2}{4\Delta x} = 0$$

$$Q_i^{n+1} = Q_i^{n-1} - \left(\frac{\Delta t}{2\Delta x} \right) \left((Q_{i+1}^n)^2 - (Q_{i-1}^n)^2 \right)$$

Higher Order Finite Difference

- Start with Taylor approximation

$$Q_i^{n+1} = Q_i^n + \Delta t \left(Q_i^n\right)_t + \frac{1}{2}\Delta t^2 \left(Q_i^n\right)_{tt} + \frac{1}{6}\Delta t^3 \left(Q_i^n\right)_{ttt}$$

$$q_t = -\left(\frac{q^2}{2}\right)_x \qquad q_{tt} = \left(\frac{q^3}{3}\right)_{xx} \qquad q_{ttt} = -\left(\frac{q^4}{4}\right)_{xxx}$$

- Put in terms of x derivatives

$$Q_i^{n+1} = Q_i^n - \frac{1}{2}\Delta t \left({Q_i^n}^2\right)_x + \frac{1}{6}\Delta t^2 \left({Q_i^n}^3\right)_{xx} - \frac{1}{24}\Delta t^3 \left({Q_i^n}^4\right)_{xxx}$$

- Uses centered finite differences on x

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{4\Delta x}(\ldots) + \frac{\Delta t^2}{6\Delta x^2}(\ldots) - \frac{\Delta t^3}{48\Delta x^3}(\ldots)$$

Finite Volume Method

Integrate both sides with respect to x:

$$q_t = -\left(\frac{q^2}{2}\right)_x \qquad \frac{d}{dt}\int_{C_i} q\,dx = \left(\frac{q^2(t, x_{i-1/2})}{2}\right) - \left(\frac{q^2(t, x_{i+1/2})}{2}\right)$$

Integrate both sides with respect to t:

$$\int_{C_i} q(x, t_{n+1})dx - \int_{C_i} q(x, t_n)dx = \int_{t_n}^{t_{n+1}} \frac{q^2(t, x_{i-1/2})}{2}dt - \int_{t_n}^{t_{n+1}} \frac{q^2(t, x_{i+1/2})}{2}dt$$

Finite Volume form:

$$\frac{1}{\Delta x}\int_{C_i} q(x, t_{n+1})dx = \frac{1}{\Delta x}\int_{C_i} q(x, t_n)dx - \frac{\Delta t}{\Delta x}\left(\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \frac{q^2(t, x_{i+1/2})}{2}dt - \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \frac{q^2(t, x_{i-1/2})}{2}dt\right)$$

Notice:

$$Q_i^{n+1} \approx \frac{1}{\Delta x}\int_{C_i} q(x, t_{n+1})dx$$

Define flux to be:

$$F(x_i, x_{i+1}) \equiv \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \frac{q^2(t, x_{i-1/2})}{2}dt$$

Finite Volume Formula

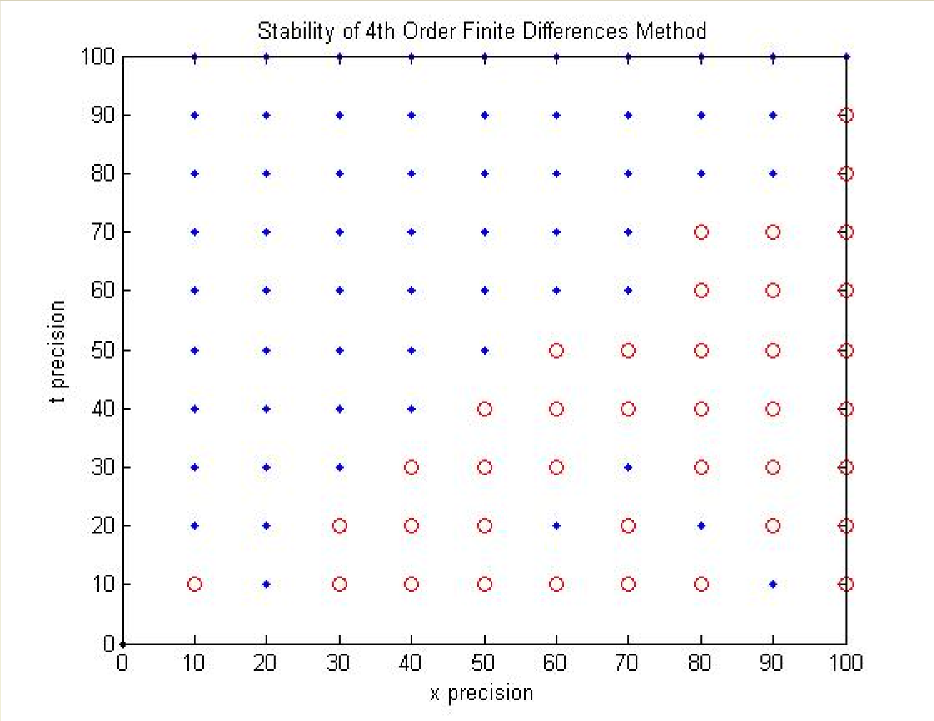
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F(x_i, x_{i+1}) - F(x_{i-1}, x_i)\right)$$

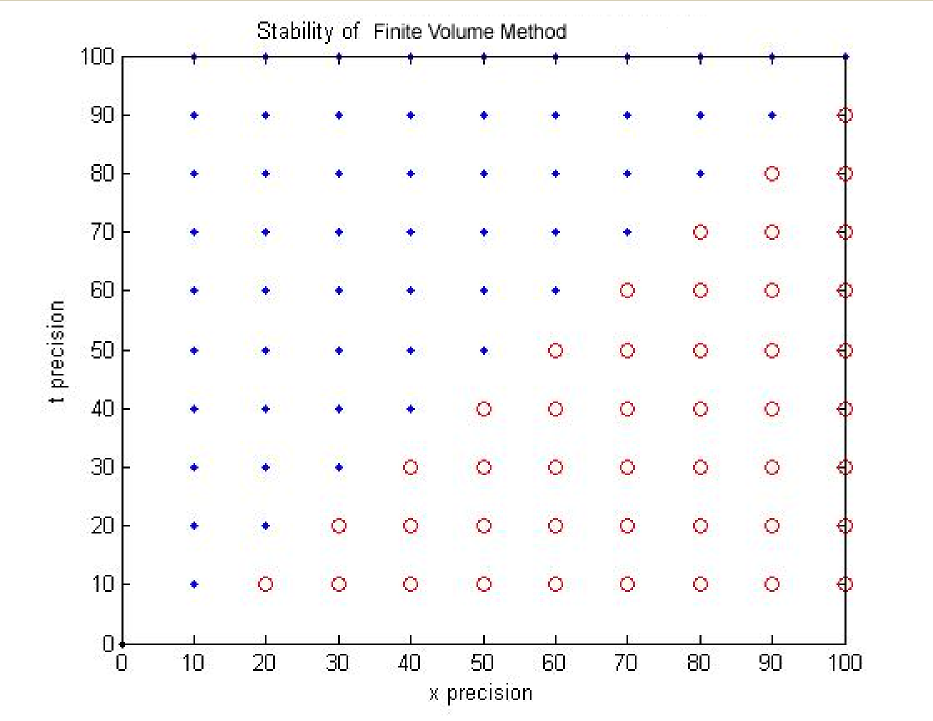
- Many ways to estimate the flux
- I used the Local Lax-Friedrichs Method
- By the Lax Wendroff Theorem this method converges to the proper weak solution as the mesh is refined

$$F(x_{i-1}, x_i) = \frac{1}{4}\left({Q_{i-1}^n}^2 + {Q_i^n}^2 - 2\max_{x \in C_i}(q(x,t_n))(Q_i^n - Q_{i-1}^n) \right)$$

Stability Criterion

The stability criterion for both appear to be approximately the same. However stability for the finite volume method is much well defined.





References

- LeVeque, Randall J., *Finite Volume Methods for Hyperbolic Problems*. Cambridge University Press (2002)
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