An Iterative Solver for the Diffusion Equation

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The Diffusion/Heat Equation

$$u_t = a + D \cdot u_{xx}$$

- u is the concentration/temperature
- a is a source/sink
- D is a diffusion/thermal diffusivity constant
- t is time, x is space

The Hitch

- For clinic, we needed arbitrary Dirichlet boundary conditions through the middle
- These BCs simply hold the concentration at a fixed amount
- Exact solution cannot be found easily

The Solution!

- Use an iterative solver
- For clinic we actually used
  - The 3-D equation
  - The Gauss-Seidel method
  - The backwards Euler FDA
  - C++
- I wanted to try
  - 1, 2, or 3 dimensions
  - Dirichlet, Neumann, and Cauchy BCs
  - The Jacobi or SSOR methods
  - The backwards Euler FDA
  - Sparse matrices in Matlab

Iterative Methods

We want to solve

$$A\vec{x} = \vec{b}$$

but we don't want to invert A (time constraints, etc). Divide up A so that

$$A = D + L + U$$

where

- D is diagonal
- L is lower triangular
- U is upper triangular

The Jacobi Method

Pick $\vec{x}^{(0)}$ to be the initial guess at a solution. Now, define

$$\vec{x}^{(i+1)} = D^{-1} \cdot (-L - U) \cdot \vec{x}^{(i)} + D^{-1} \cdot \vec{b}$$

If $\left|\vec{x}^{(i+1)} - \vec{x}^{(i)}\right|$ isn't small enough, repeat.

The Successive Over-Relaxation (SOR) Method

We will solve

$$\omega A\vec{x} \;=\; \omega\vec{b}$$

Noting that

$$\omega A \;=\; (D + \omega L) + (\omega U - (1-\omega)D),$$

we now have that

$$\vec{x}^{(i+1)} \;=\; (D+\omega L)^{-1} \cdot \left((-\omega U + (1-\omega)D)\vec{x}^{(i)} + \omega\vec{b}\right)$$

The backwards version is

$$\vec{x}^{(i+1)} \;=\; (D+\omega U)^{-1} \cdot \left((-\omega L + (1-\omega)D)\vec{x}^{(i)} + \omega\vec{b}\right)$$

The Symmetric Successive Over-Relaxation (SSOR) Method

We do one forwards SOR step followed by one backwards SOR step:

$$\begin{aligned} \vec{x}^{(i+1/2)} &= (D + \omega L)^{-1} \cdot \left((-\omega U + (1-\omega) D) \vec{x}^{(i)} + \omega \vec{b} \right) \\ \vec{x}^{(i+1)} &= (D + \omega U)^{-1} \cdot \left((-\omega L + (1-\omega) D) \vec{x}^{(i+1/2)} + \omega \vec{b} \right) \end{aligned}$$

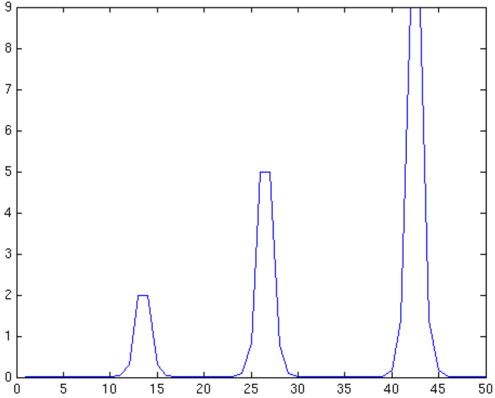
Progress So Far...

Slower than I expected

- 1-D case implemented for both Jacobi and SSOR methods with any Dirichlet BCs
- Neumann (and therefore Cauchy) BCs are not well defined in arbitrary locations, especially in a 1-D case
- Uses only sparse matrices
- Checks for stability

Oddly enough, the Jacobi method seems to converge more quickly than the SSOR!?

An Example Run...



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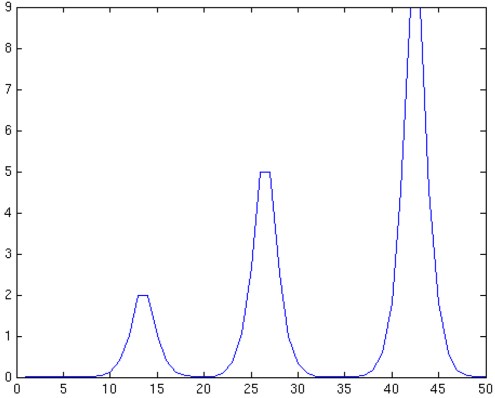

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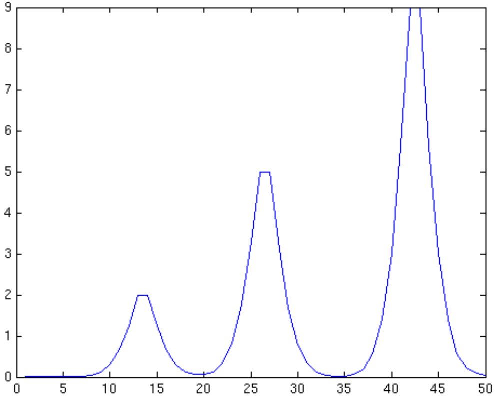

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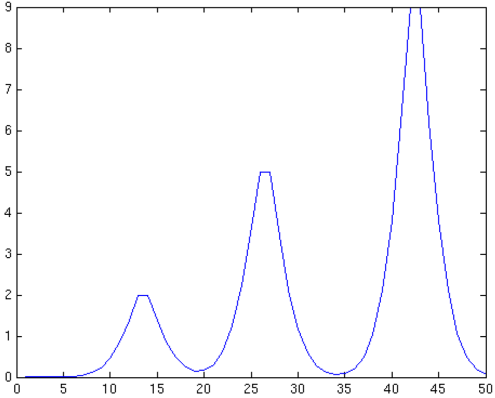

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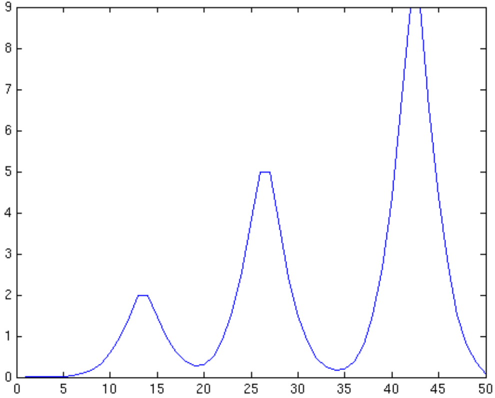

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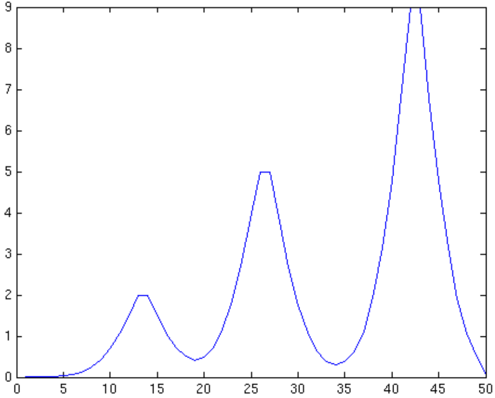

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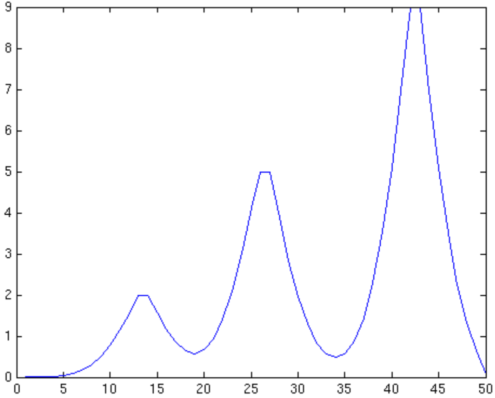

References Used

Saad, Yousef. Iterative Methods for Sparse Linear Systems SIAM, 2000.

http://mathworld.wolfram.com

That's All, Folks!

Questions?