The Chaotic Waterwheel: Exploring the Lorenz Equations

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Math 164

Final Project

Background

- Discovered in 1963 by Ed Lorenz
- Simple model of convection in atmosphere
- First showing of strange attractor and chaos
  - No fixed points
  - No periodic orbits
  - Solutions do not → infinity with time

Derivation

NO! But…

- Conservation of Mass
- Torque Balance
- Amplitude Equations

$$\dot{a}_1 = \omega b_1 - K a_1$$

$$\dot{b}_1 = -\omega b_1 - K\omega + q_1$$

$$\dot{\omega} = (-\nu\omega + \pi g R a_1) / I$$

$K = \text{Leakage Rate}$

$q_1 = \text{Inflow Rate}$

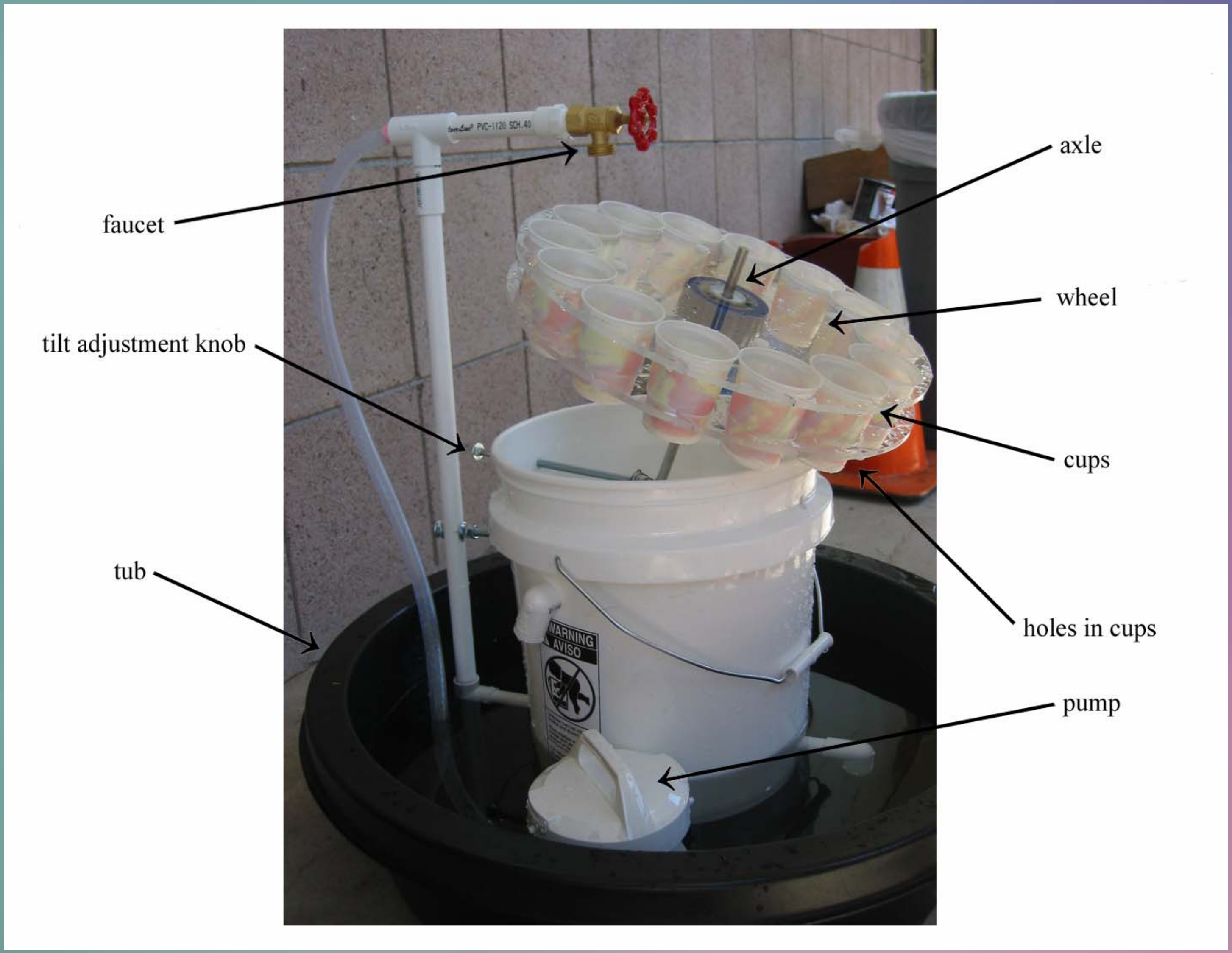
$\nu = \text{Rotational Damping Rate}$

$g = \text{Gravity (Variable)}$

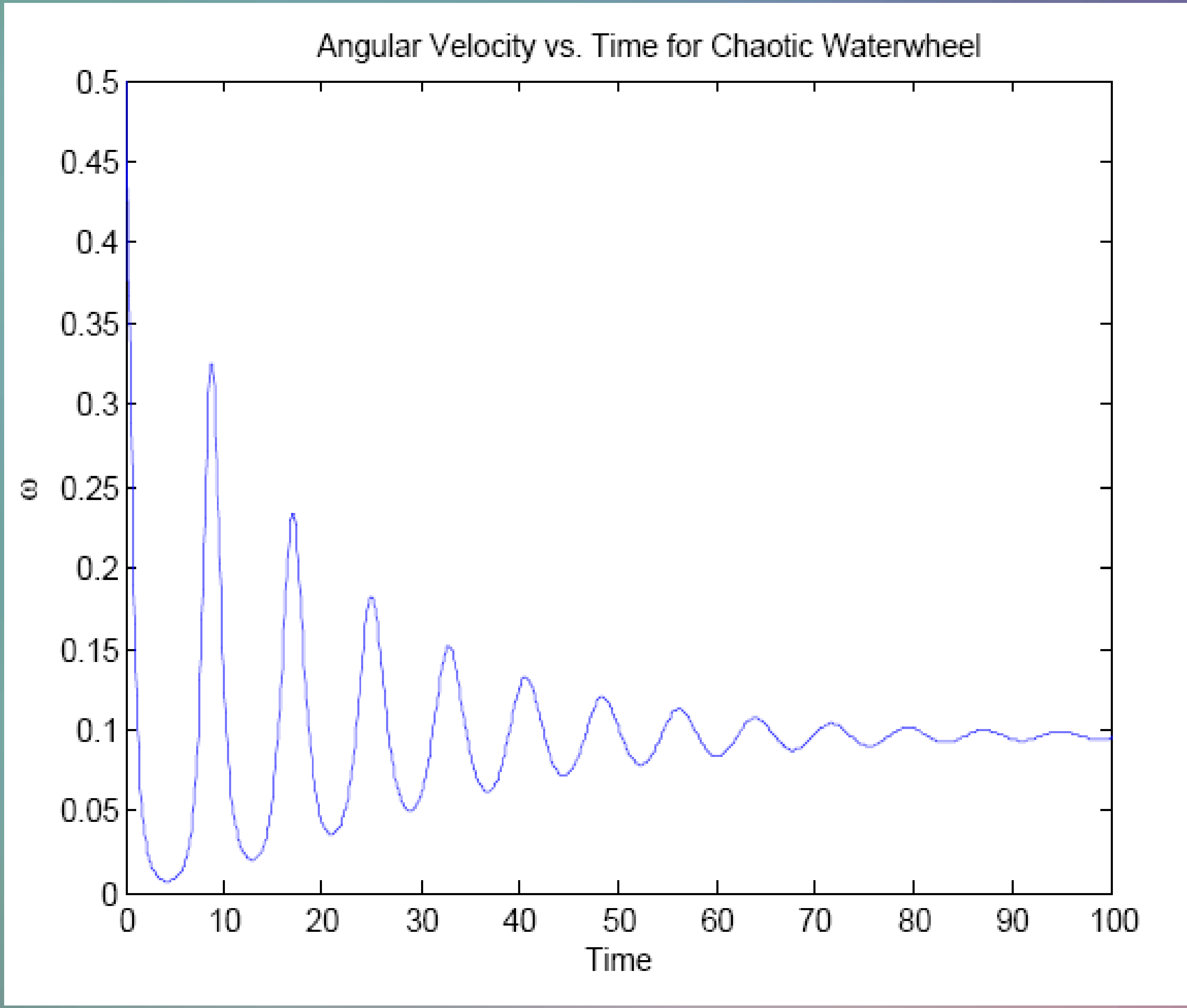
$R = \text{Radius of Wheel}$

$I = \text{Moment of Intertia}$

The Waterwheel



Animations and Results



The Lorenz Equations

Just a change of variables away!

$$\frac{dx}{dt} = \sigma(y-x)$$

$$\frac{dy}{dt} = rx-y-xz$$

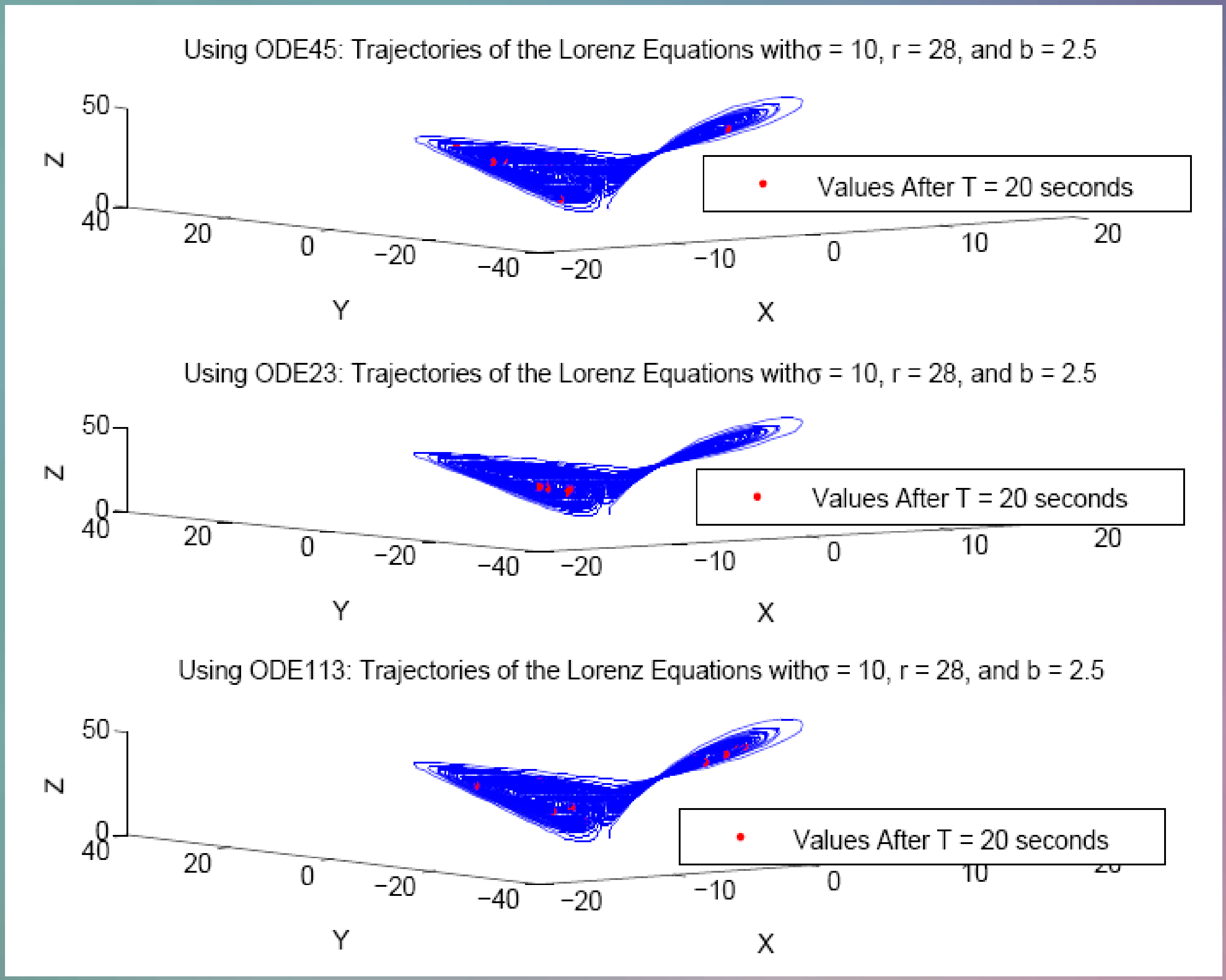
$$\frac{dz}{dt} = xy-bz$$

σ = Prandtl Number

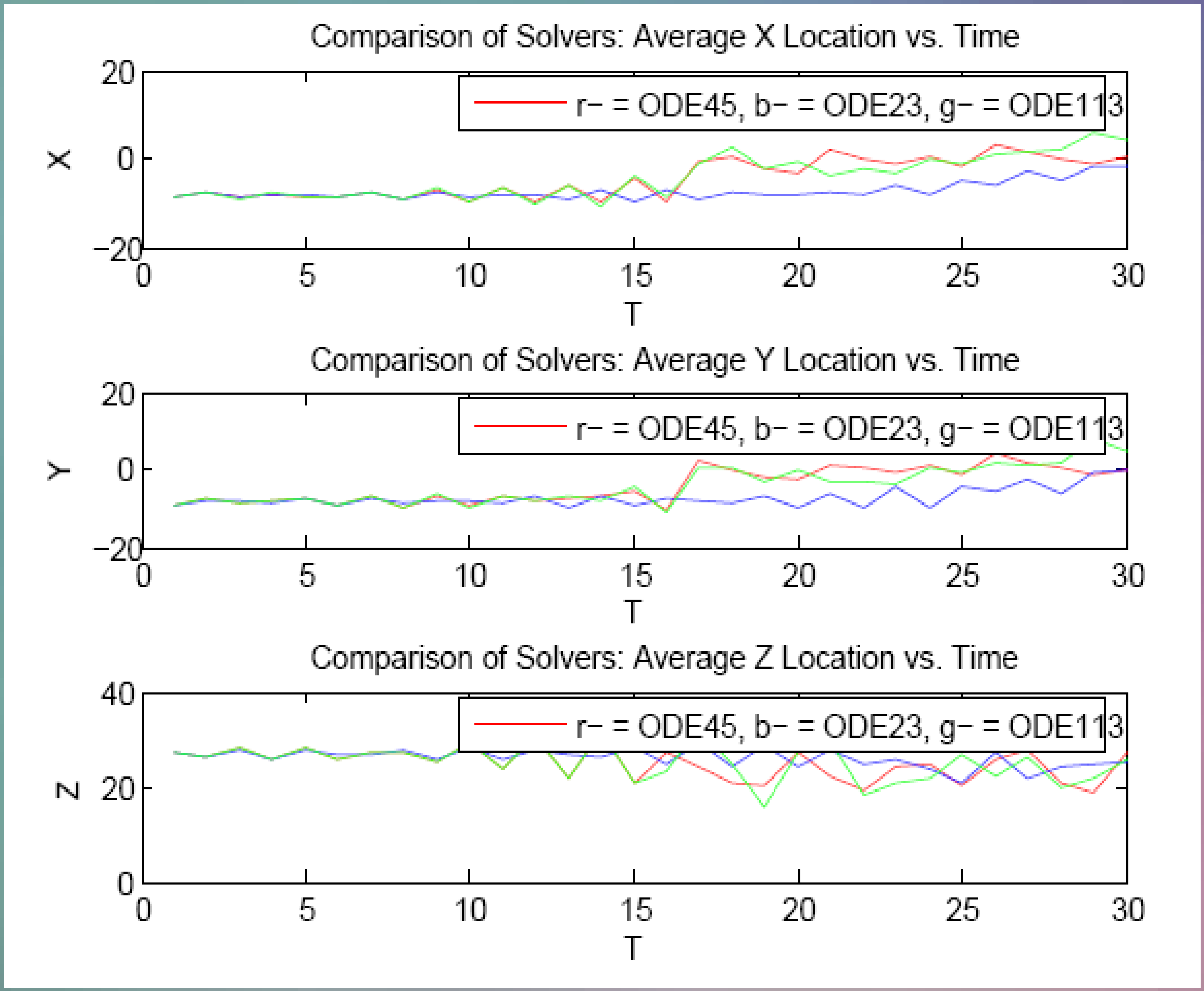
r = Rayleigh Number

b = No Name

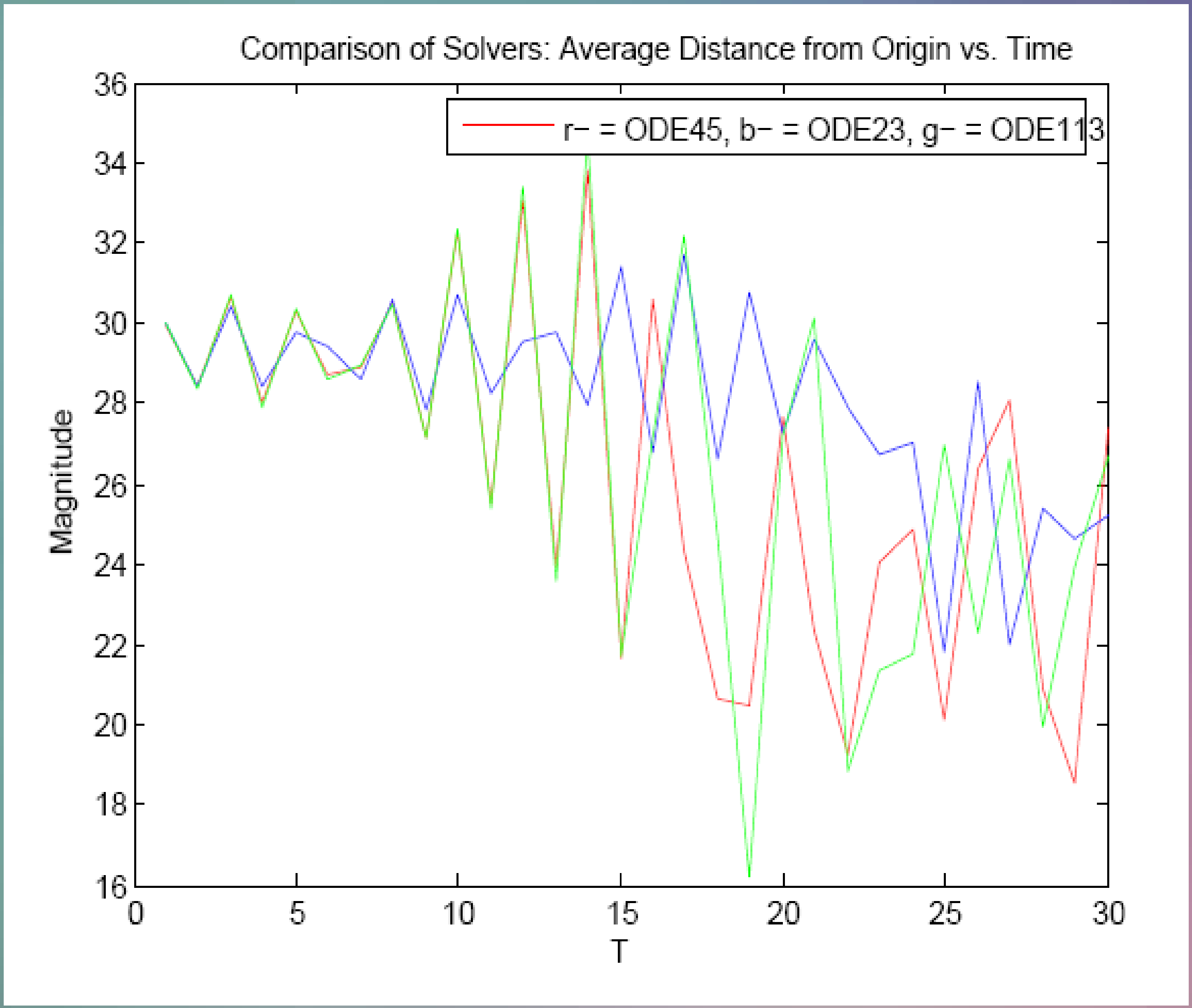
Solution Reliability



Solution Reliability

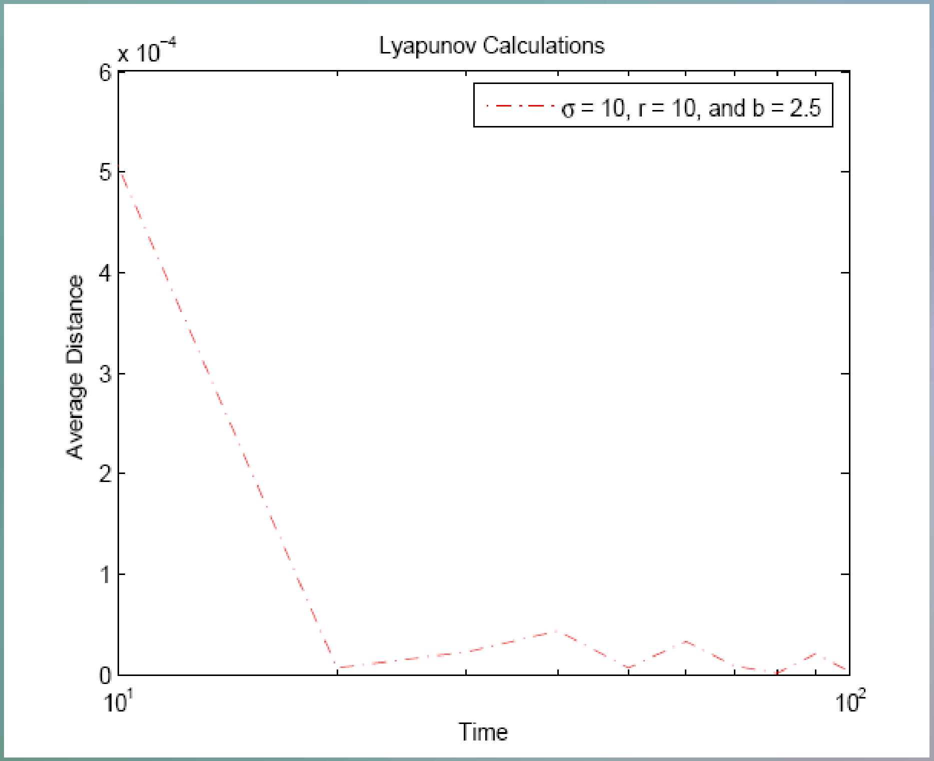


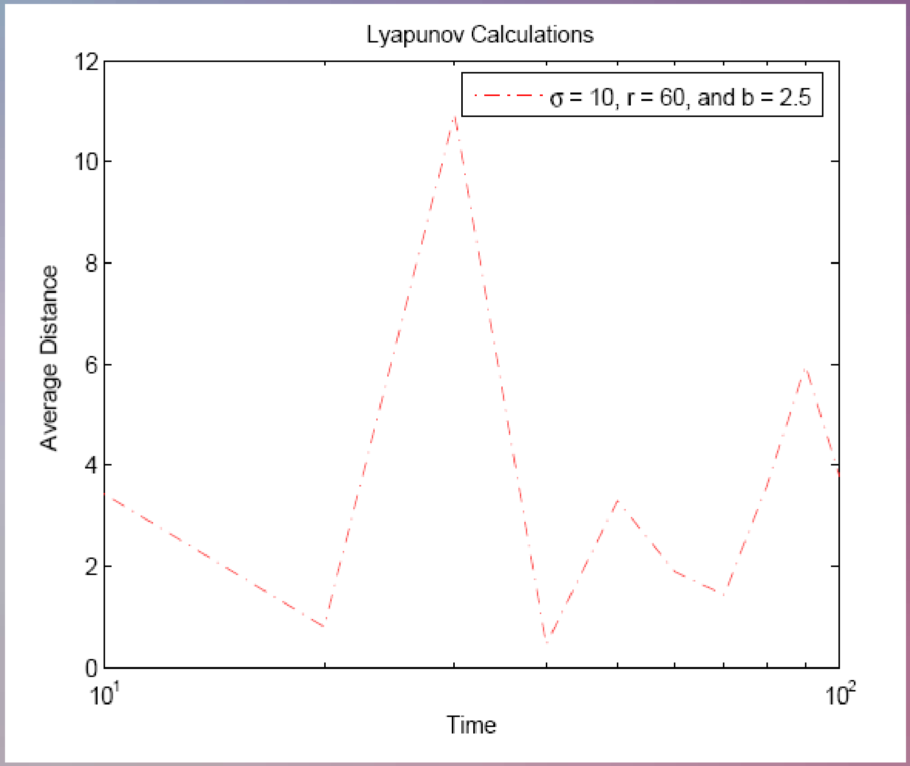
Solution Reliability



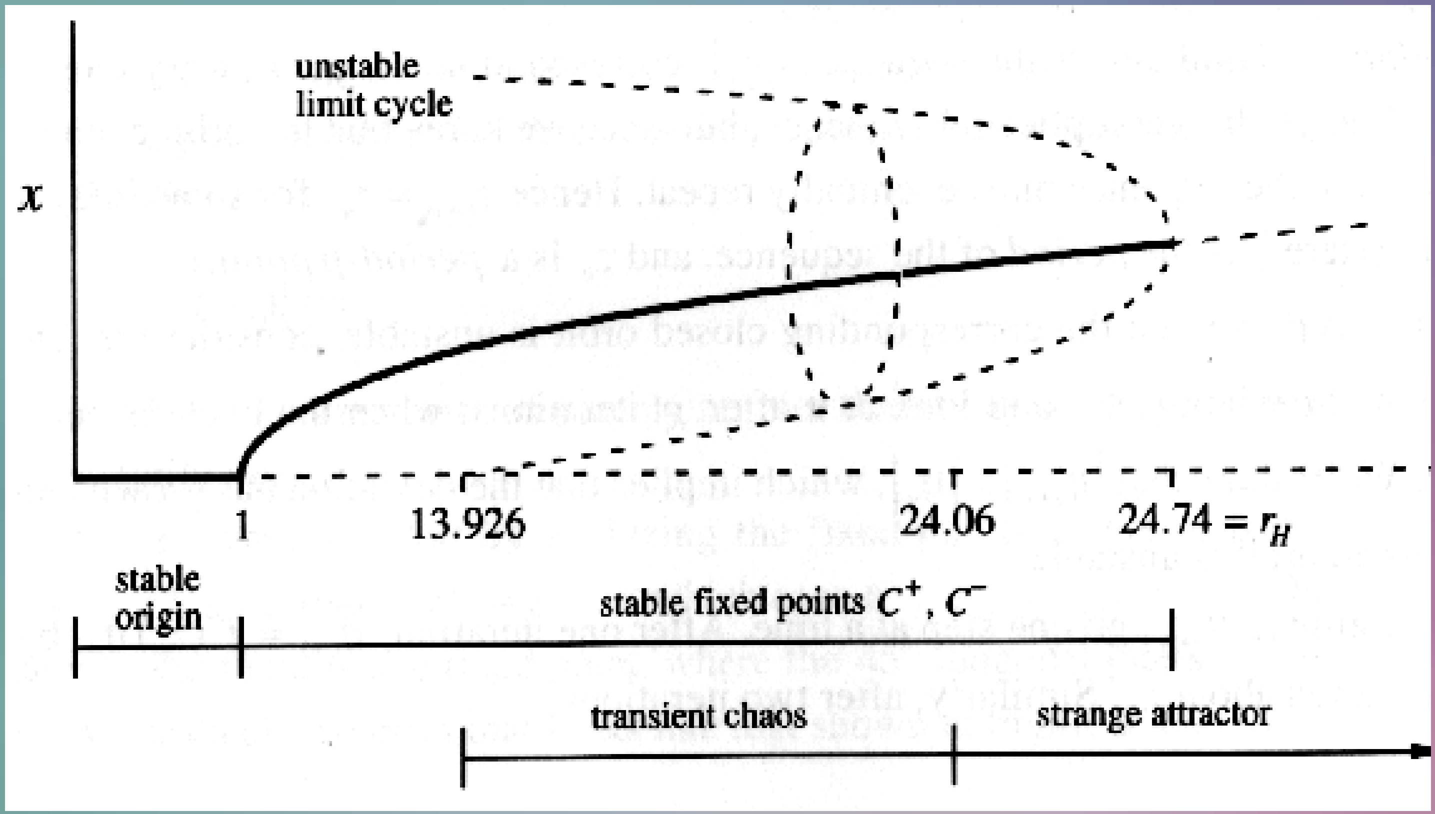
Liapunov Functions

Measure Divergence of Nearby Trajectories with Increasing Time

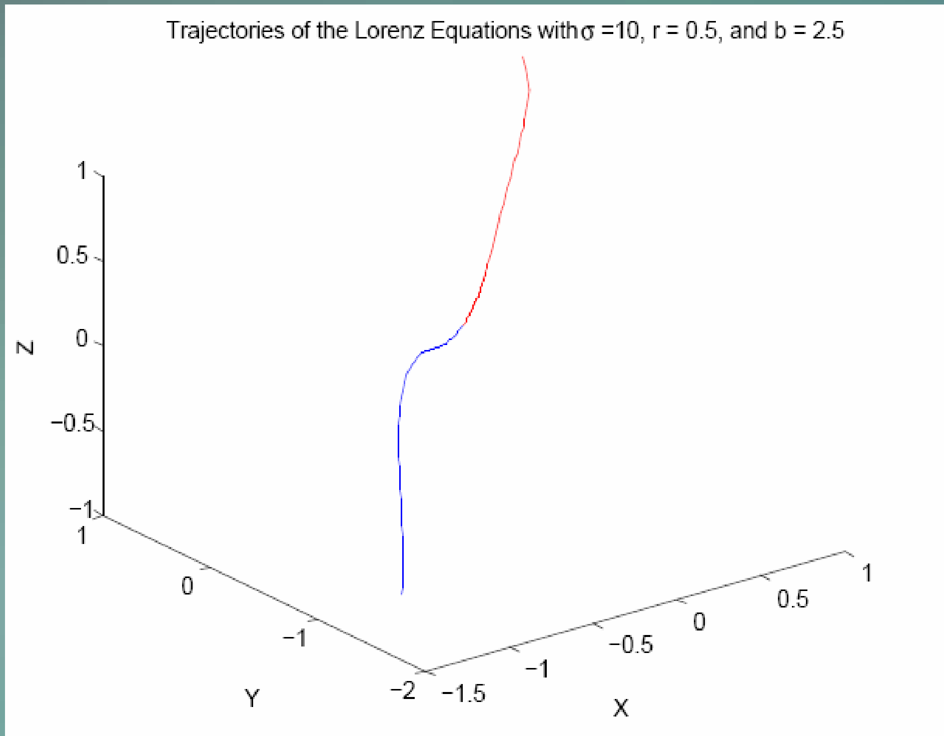


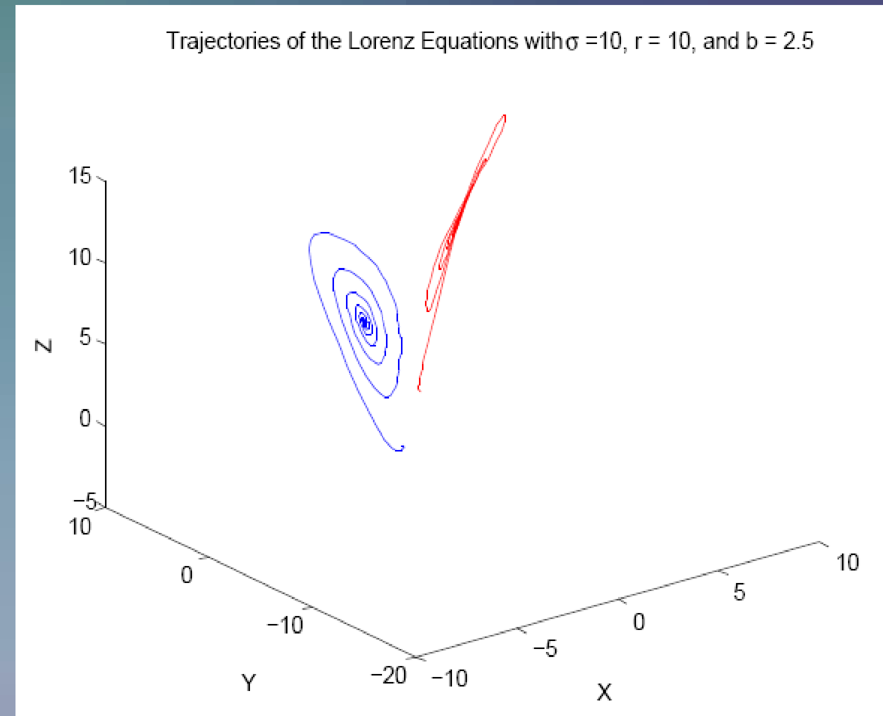


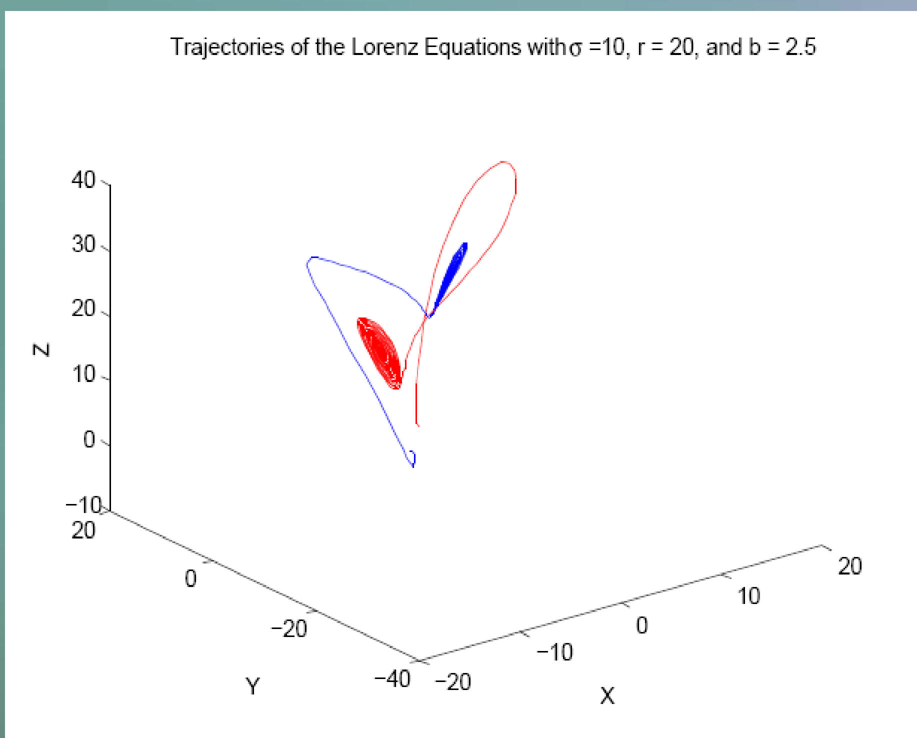
Behaviors

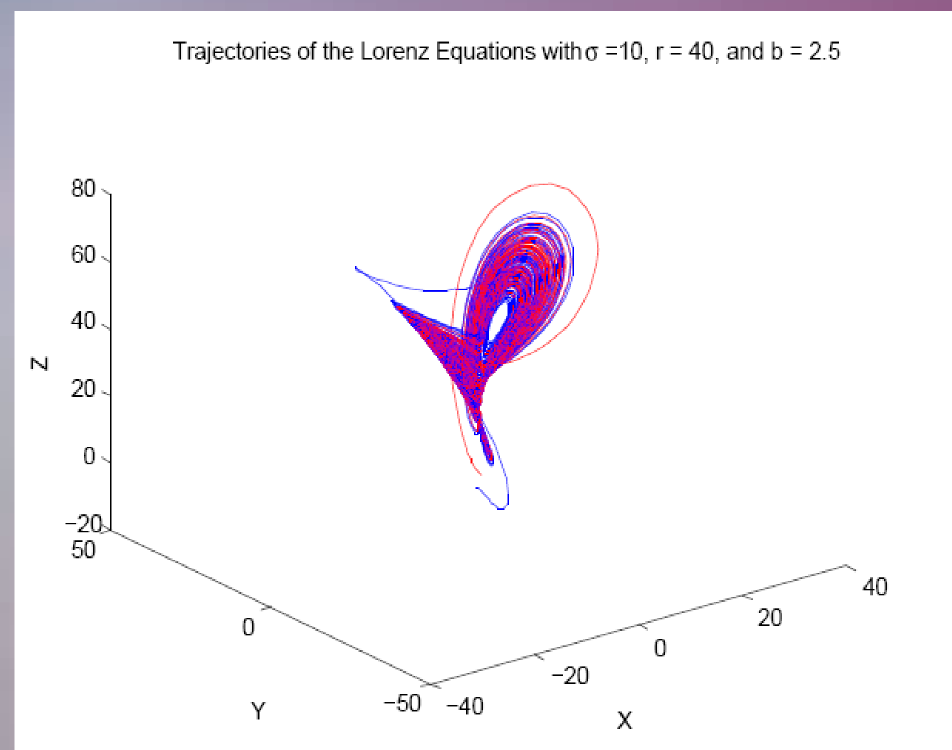


unstable
limit cycle
x
1
13.926
24.06
$24.74 = r_H$
stable
origin
stable fixed points C^+, C^-
transient chaos
strange attractor

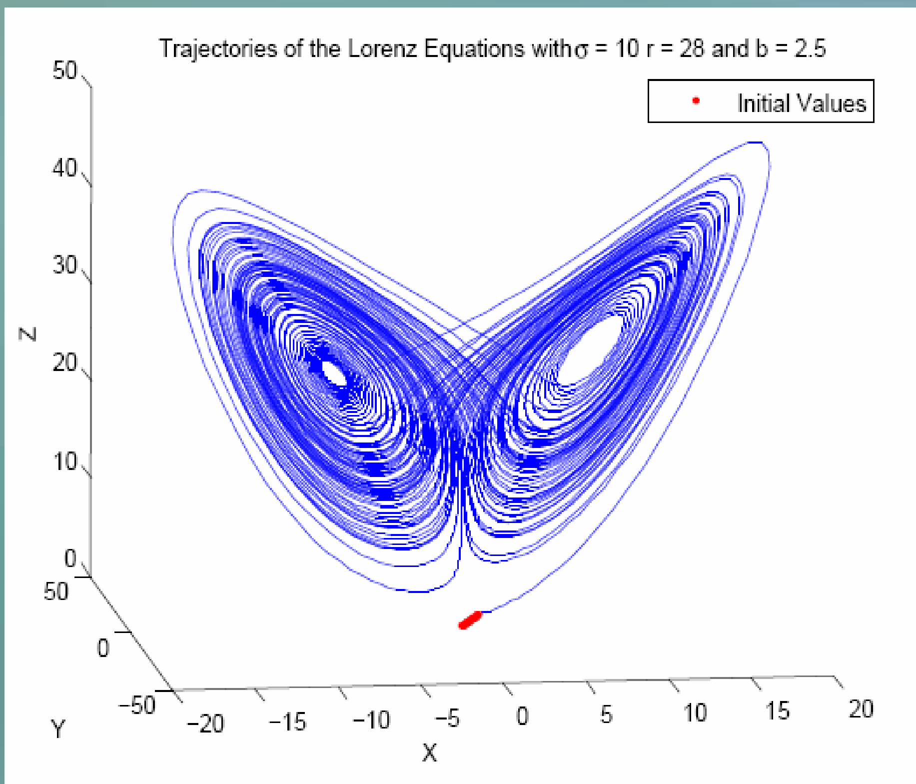

Trajectories of the Lorenz Equations with σ =10, r = 0.5, and b = 2.5
Z
1
0.5
0
−0.5
−1
Y
1
0
−1
−2
X
−1.5
−1
−0.5
0
0.5
1

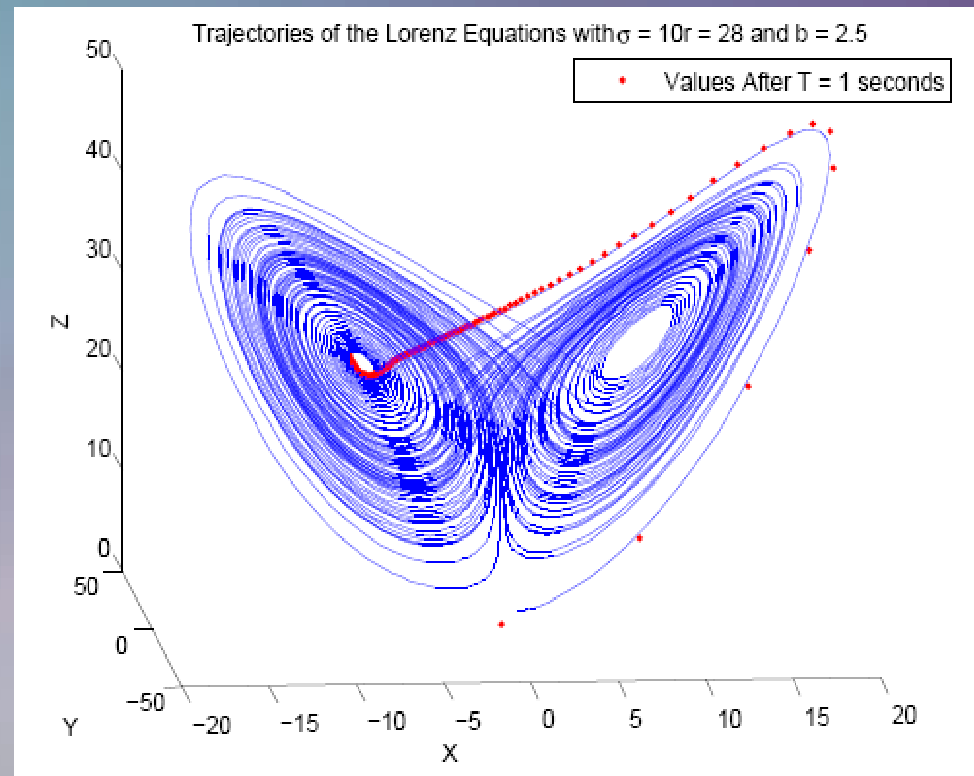

Trajectories of the Lorenz Equations with σ =10, r = 10, and b = 2.5
Z
15
10
5
0
−5
Y
10
0
−10
−20
X
−10
−5
0
5
10


Trajectories of the Lorenz Equations with σ =10, r = 20, and b = 2.5
Z
40
30
20
10
0
−10
Y
20
0
−20
−40
X
−20
−10
0
10
20

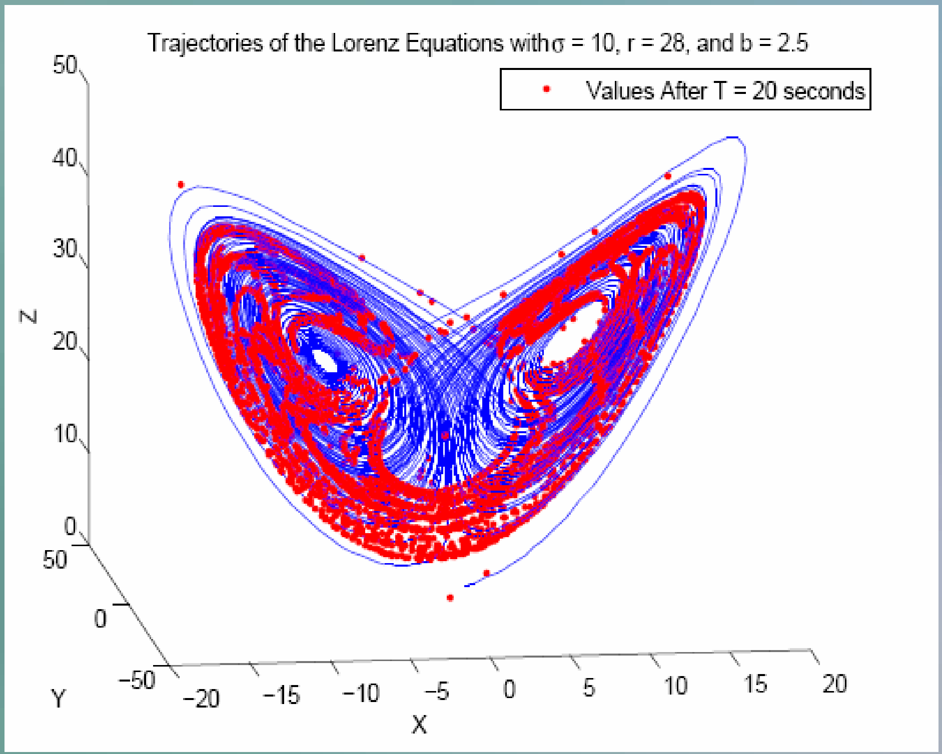

Trajectories of the Lorenz Equations with σ =10, r = 40, and b = 2.5
Z
80
60
40
20
0
−20
Y
50
0
−50
X
−40
−20
0
20
40

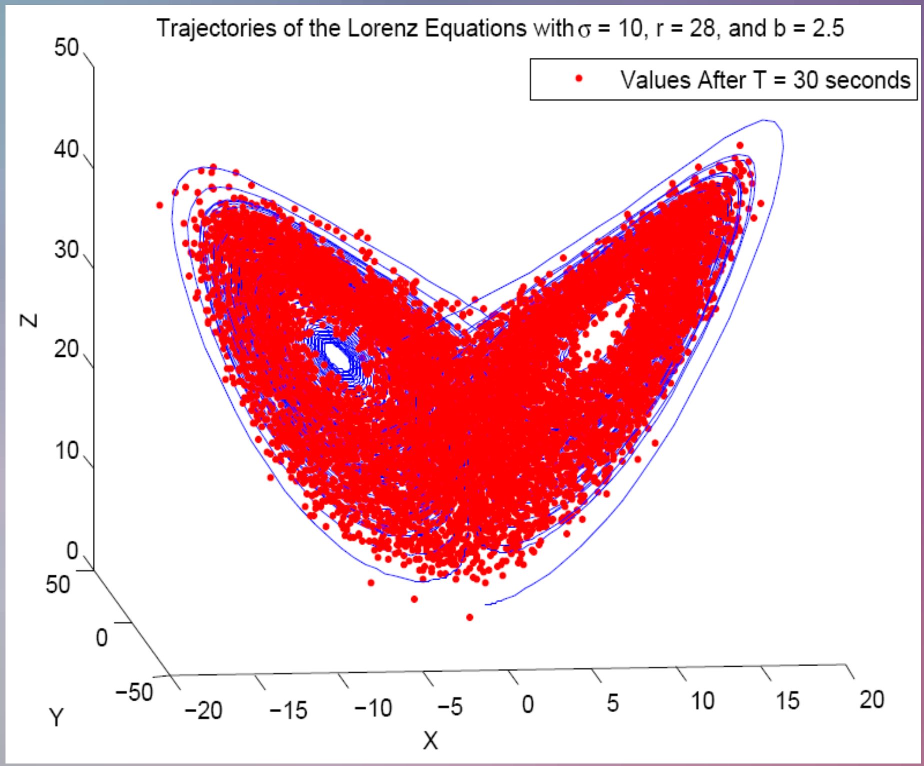
Chaos and Sensitive Dependence





Chaos and Sensitive Dependence





Left or Right Brain?