Perturbations of a Hanging Chain

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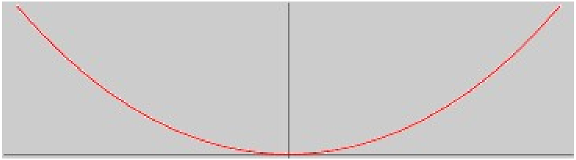
The Problem

- Model small perturbations of a chain hanging by two ends.

Shape

- A chain hanging by two ends has the shape of a catenary



Parameterization of a Catenary

- Use arc-length parameter s, with $s = 0$ corresponding to the center of the catenary. The constant α determines the overall curvature of the catenary

$$x(s) = \alpha \sinh^{-1}\left(\frac{s}{\alpha}\right),$$

$$y(s) = \alpha \cosh\left(\sinh^{-1}\left(\frac{s}{\alpha}\right)\right)$$

- Our chain has arc length 2, so $-1 \leq s \leq 1$.
- Also, we assume that $\alpha = 1$.

Assumptions in our Model

- The chain will behave like a heavy, inextensible (non-stretching) rope.
- We will only consider perturbations of the chain from its resting state.
- Extremely high modes of motion will contribute negligibly to the motion of the chain.

The Governing Equations

- There is no motion in the x or z direction!
- The PDE which rules the system:

$$y_{tt} = g\left(y_s\sqrt{1+s^2}\right)_s, -1 \leq s \leq 1, t > 0$$

- Subject to the following conditions:

$$y(1,t) = y(-1,t) = y_t(s,0) = 0$$

$$y(s,0) = f(s)$$

The Governing Equations (Continued)

- Using separation of variables, this is an eigenvalue problem!
- The DE for $S(s)$ determines the eigenvalues and eigenfunctions:

$$S_{ss} + \frac{sS_s}{(1+s^2)} + \frac{\lambda S}{\sqrt{1+s^2}} = 0$$
$$-1 \leq s \leq 1$$

- Subject to:

$$S(-1) = S(1) = 0$$

- The DE for $T(t)$ reduces to:

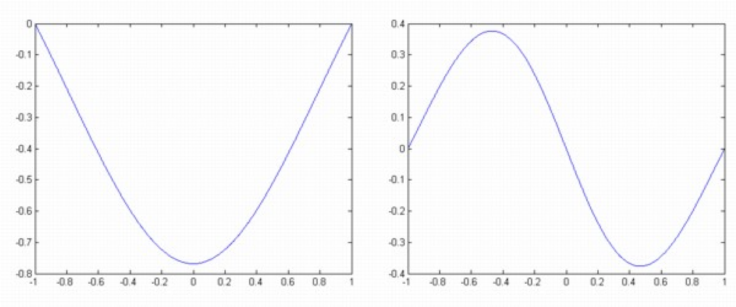
$$T(t) = B\cos\left(t\sqrt{g\lambda}\right)$$

Strategy for the Numeric Solution

- Find a few eigenfunctions and eigenvalues of $S(s)$.
- Determine the B_n for each eigenvalue to meet the initial condition $f(s)$.
- Animate the result!

Numerical Solutions for the Modes

- Using MATLAB's bvp4c command, we solved for the eigenfunctions, which represent the basic modes of chain motion.



The first two eigenfunctions, with corresponding $\lambda \approx 3, 11$

The Motion of the Chain

- MATLAB demo of results.

References:

- Yong, Darryl. “Strings, Chains, and Ropes”, SIAM Review (to appear), 2006.
- Eric W. Weisstein. "Catenary." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Catenary.html

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