Dynamics of Mobile Toroidal Transformer Cores

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Abstract

A simplistic model of a c-core transformer will not accurately predict the output voltage. The inclusion of the magnetization of iron and an air gap reduces the maximum error by 72 standard deviations. They are the most important factor to include in any model of similar situation. Allowing the cores to move recreates some interesting physical effects, but it is difficult to calibrate and not the major factor in the error. This model shows the behavior of the transformer is dictated by the number of turns of wire on the core. The cores have two behaviors. With a low number of turns, the cores are connected for the majority of the time. At a high number of turns, the cores remain a fixed distance apart. Studying this system in depth has given us a better understanding of why the system behaves as it does.

Introduction

The goal of the Southern California Edison clinic team is to create novel methods for extracting electrical power from electrical power lines. One proposed solution is a "clamp-on" toroid. This device uses the changing magnetic fields created by the power lines to create voltage. To obtain an understanding of the core's functionality, the SCE team created a simple model of the system. Unfortunately, it is not an accurate model. One prediction has an error of 80 standard deviations compared to the measured value. While it was only 20% away it implies we do not understand the dynamics of this system. The error stems from two majors assumptions. The first assumption is that a c-core has a constant magnetic permeability. Since the cores are produced from a proprietary alloy of iron, the magnetic permeability changes with the magnetic field. The relationship between the two is the metal's magnetization curve. The next assumptions it that the transformer is one continuous piece. It is actually comprised of two separate toroidal core halves, also known as c-cores. Experimentation shows as the cores are pulled apart, increasing the air gap between them, the output voltage decreases. Assuming the air gap is negligible is another factor contributing to the overestimation. To account for these assumptions, the Fixed Distance model was created.

Questions for the Fixed Distance Model

The goal of the Fixed Distance model is to determine whether:

- The magnetization curve deforms the voltage waveform or some other effect?
- The magnetization curve and the air gap account for the majority of the error in the team's model?

The Fixed Distance model itself has one serious assumption. It does not allow the c-cores to move. Physically, it is inconceivable that two free bodies will remain still. To account for this the Variable Distance model was created. The Variable Distance model allows one c-core freedom of motion while the other oscillates at the same frequency as the power line it is mounted on. This model cannot replace the Fixed Distance model because the equations of motion for a pair of c-cores depends on many parameters which we do not have data on. Therefore, this model will only be used to answer qualitative questions since the system cannot be calibrated with the data on hand.

Questions for the Variable Distance Model

The Variable Distance Model will determine:

- How does the motion of the cores depend on N?

- Is the Fixed Distance model's assumption the cores remain a fixed distance apart a good one?

Numerical Models

As with all physical models, simplifying assumptions are necessary. Both the Fixed and Variable models assume:

- The C-cores are identical
- No magnetic flux is lost
- Friction is the only cause of energy loss
- The cores cannot rotate

These assumptions are common in engineering, but they are sources of error in this model. The physical shape of the "clamp-on" toroid is different from a traditional transformer. However, it can be represented as a traditional transformer as shown in Figure 1.

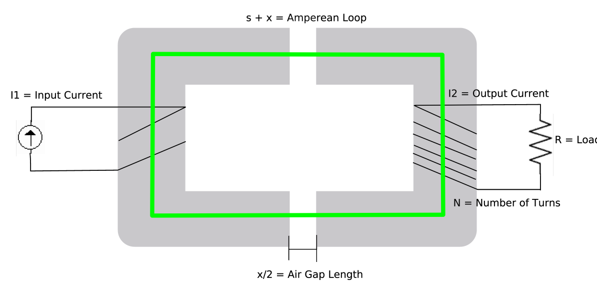

Figure 1: This is the system being modeled. I_1 is the current in the power line. It drives the entire system. I_2 is the induced current through the load. It determines the output voltage by $V = I_2R$. N is the number of turns on the core, A is the cross sectional area and s is the magnetic path length. x, the air gap, is fixed in the Fixed Distance model, but allowed to vary in the Variable Distance Model.

N is the number of turns of wire around one of the c-cores, I_1 is the input current, and I_2 is the output current. The output voltage is related to that value by $V = I_2R$. x is twice length of the air gap, A is the magnetic cross sectional area, and s is the path length. Since A and s depend on the geometry of the c-core and the team only took extensive data on one c-core, in order to compare with measured data they could not be varied. Since their effects can not be verified, they will be held constant in all models. However, there is significant data on the effects of N, so its effects will be the focus of this project.

The governing equations for the system shown in Figure 1 are easily found, but evaluating them requires substantial preparation. First, a model for the magnetization curve of the iron core needed to be created. This is important since it is included in the governing equation, so we must be able to evaluate it at any point. Unfortunately, research conducted both for clinic and for this project were not able to find any deterministic models of a ferromagnet. Since the magnetization curve needs to accurately reflect the data provided by Metglas, the manufacturers of the core, stochastic models such as the Ising Model were rejected because they are too difficult to calibrate.

To provide an equation to fit the data, a Piecewise Hermite Interpolating Polynomial(PCHIP) was employed. The results of the interpolation are shown below.

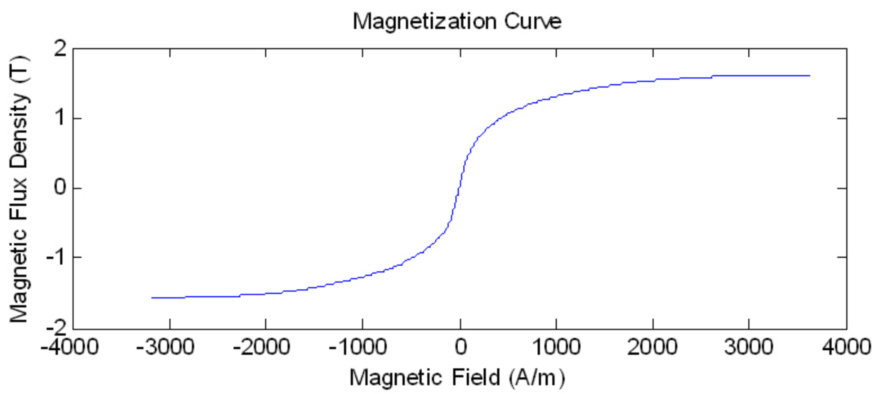

Figure 2: This is the magnetization curve interpolated with MATLAB's PCHIP method. This method was chosen because it is less oscillatory than a cubic spline. It is important to reduce oscillations because the function's derivative is needed.

The PCHIP method was chosen over a cubic spline because it is less oscillatory. There is no harm in reducing oscillations since the actual curve does not oscillate. The PCHIP method has a continuous derivative as well, which is ideal since the derivative of this function must be taken. The polynomial nature of this interpolant allows the derivative to be taken analytically. Thus, the only source of error it the interpolant itself. Unfortunately, this is only valid when there is no air gap between the cores.

Accounting for the effects of an air gap is computationally expensive but simple. The only assumption is the magnetic flux cannot be lost traveling through the air gap. Therefore, $B_{\text{in metal}} = B_{\text{in air}}$. From Ampere's Law,

$$\oint \vec{H} \bullet d\vec{l} = I_1 + NI_2.$$

Integrating along the Amperean loop, highlighted in green in Figure 1, gives

$$B = \frac{\mu_0}{x}\left(I_1 + NI_2 - Hs\right) = f(H)$$

where $f(H)$ is the magnetization curve in Figure 2. This gives H as a function of I_1, I_2, x, and N. Plugging H into the magnetization curve, $f(H)$, gives B as a function of those same variables. Solving the differential equation requires knowing this curve continuously within a certain domain. This requires interpolation with respect to two variables. To obtain the interpolating surface, a root finding method was employed at different values of x and $(I_1 + NI_2)$. MATLAB's interp2 method was used to obtain a surface.

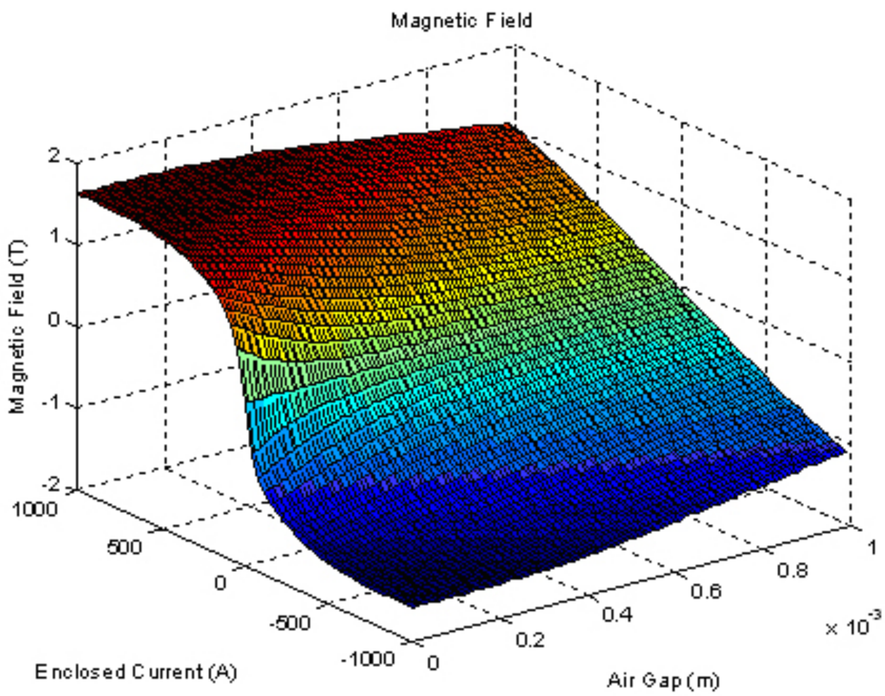

Figure 3: This is the interpolated surface which results from root finding for many values of x and $I_1 + NI_2$. While this step is not strictly necessary, running a root finding method within a stiff ode solver proved to be far too computationally intensive. This allows values to be precomputed to save on time.

Finding this surface is not strictly necessary, but running a root finding method within a finite difference method proved extremely computationally expensive. Creating this interpolating surface allows the relation between the variables to be precomputed, saving time while solving the differential equation. With these interpolants constructed, it is now possible to solve the governing equations for this system.

Fixed Distance Model

The Fixed Distance model assumes the cores do not move relative to each other. The derivation comes entirely from Faraday's Law, so

$$V = I_2 R = -NA\frac{dB}{dt}.$$

Expanding the derivative gives

$$V = I_2 R = -NA\left(\frac{df}{dZ}\right)\left(\frac{\partial I_1}{\partial t} + N\frac{\partial I_2}{\partial t}\right), \quad (1)$$

where $\frac{df}{dH}$ is the derivative of Figure 2 and $Z = I_1 + NI_2$. Since these variables are known, $\frac{\partial I_2}{\partial t}$ can be isolated. To solve differential equation, a finite difference method, MATLAB's ode15s, was chosen because the solution is of the form $Ae^{-Ct} + G(t)$. This means it is stiff. Any non-stiff finite difference method would have its error compound too quickly to give accurate results. In addition, ode15s is the most accurate of MATLAB's stiff ode solvers. Since this is an IVP, more accurate methods like a spectral method are not feasible. A finite difference one is the only method that really applies to this situation.

This model is a significant improvement over the previous model. Verifying this model is done by ensuring it acts as an ideal transformer at low input currents. This behavior was observed experimentally and is mentioned in introductory Electricity and Magnetism texts. If the input current is a sinusoid, the output current and therefore voltage are sinusoids too. These results are observed below,

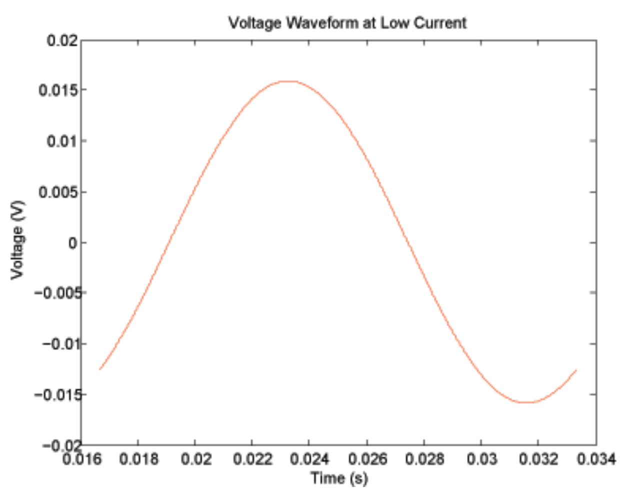

Figure 4: This is the output voltage waveform when the input current is a small amplitude sinusoid. As expected, the output waveform is a sinusoid too.

As expected, the output voltage is a sinusoid. This was repeated for other input waveforms and the results were similar. It is safe to say at low input currents this does act as an ideal transformer. Now the Fixed Distance model will be used to examine larger currents, which provide more interesting results.

Fixed Distance Model Results

The model will not simulate the output voltage waveform with a high amplitude input current. We expect the output voltage to become more peaked, like the following figure in red, but we did not know what the cause was.

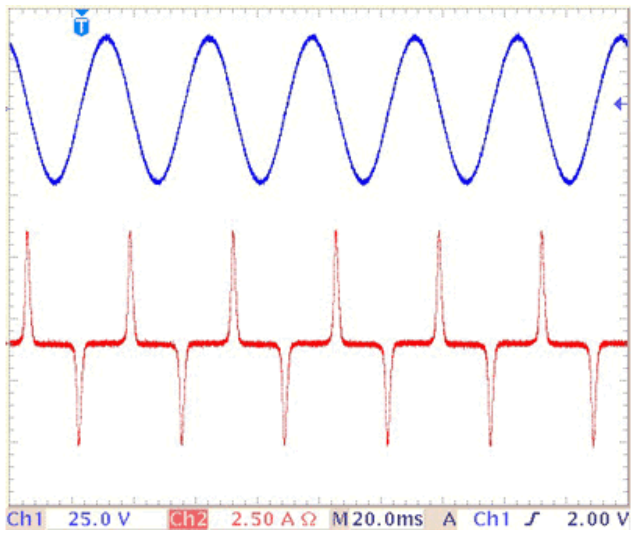

Figure 5: This is data taken from a saturating transformer in an audio amplifier. The input voltage is in blue and the output voltage in red.

This is experimental data gathered from the saturation of an audio transformer by the Rane company.[1] While the circumstances are different, the physics is the same. The waveform in red is the shape of the output voltage. This model predicts a waveform of the following shape,

[1]Mathews, Paul. "Unwinding Distribution Transformers". http://www.rane.com/note159.html. 20 April 2006.

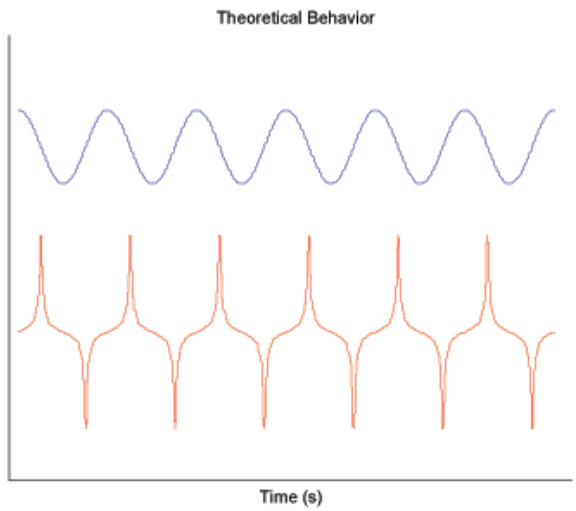

Figure 6: This is predicted output waveform of the c-cores with a large input current. The input current is in blue and the output voltage in red.

Figure 6 and Figure 5 are very similar. Both exhibit the same deformation at the same time. This effect appeared before the air gap was accounted for. Therefore, the deformation is caused solely by the magnetization curve of the metal.

The last test is to compare the root mean square voltage to the data the team collected. While the match is not perfect, it shows this model is fairly accurate.

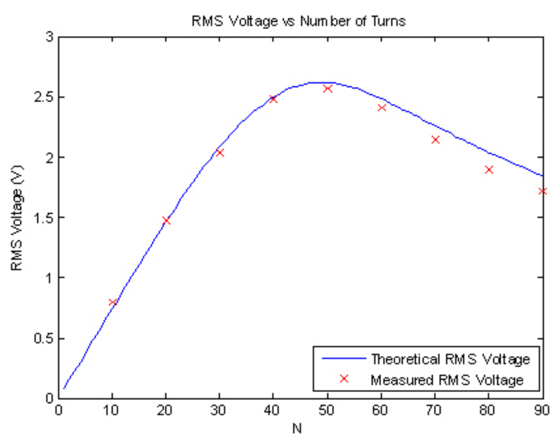

Figure 7: This is a comparison of the theoretical RMS voltage and the measured RMS voltage. The fit was calibrated at small N, and as N grows, the error systematically grows as well.

Figure 7 compares the measured voltage to the predicted voltage. This fit is not perfect, at $N = 90$ the measured and theoretical values are 8 standard deviations apart.

Fixed Distance Analysis

The fixed distance model fixes the majority of the problems in the previous model. At $N = 90$, it overestimates the RMS Voltage by 8 standard deviations. This is a significant improvement on the 80 standard deviation overestimation at that same point by the old model. This model only accounts for an air gap and magnetization. They must be the cause of this increase in accuracy. In addition, we begin to understand the trade off between N and voltage. At small N, the magnetic field is large. This can be seen because the voltage waveform is deformed as in Figure 6. Since the magnetization curve is the cause of the deformation, we can imply a deformation implies a very large magnetic field. This is because at small currents and therefore small magnetic fields, the transformer acts ideally. A larger current means a larger magnetic field, which is when the waveform changes. A large magnetic field implies a large change in magnetic field. Since the output voltage is related to the rate of change of the magnetic field, a small N appears to be beneficial by ensuring a large magnetic field.

However, as with a normal transformer, the larger N, the more voltage a transformer extracts. To have a large changing magnetic field, we must give up the ability to extract voltage from that field. If N is too large, then the magnetic field will be small. This is seen in Figure 4. The output voltage waveform is not deformed at all. That means the core is not saturated and dealing with a relatively small magnetic field. The overall effect is evident in Figure 7. To get maximum voltage, we must balance these effects and select a N which is somewhere in between. From this simple model, it is already possible to conclude that the magnetization curve and air gap play critical roles in this system. Neglecting them is impossible. In addition, the most important variable in obtaining the maximum output voltage is N. The other variables will always increase or decrease the voltage, but N needs to be selected properly.

Variable Distance Model

The variable distance model assumes one core vibrates and the other core is free to move. The core that vibrates does so because it is in contact with the power line. That line vibrates at $2f$, twice the frequency of the current in the line. Like all vibrations, it also has higher harmonics with lower amplitudes. The other core is free to move, though it is influenced by friction and the magnetic force. The governing equations are,

$$\begin{aligned} I_2 R &= -NA\left(\frac{df}{dZ}\right)\left(\frac{\partial I_1}{\partial t} + N\frac{\partial I_2}{\partial t}\right) && (2) \\ \dot{x} &= 2v && (3) \\ \dot{v} &= -\frac{B^2 A}{2\mu_0} - \frac{v}{|v|}\mu_k g && (4) \end{aligned}$$

where v is the velocity, μ_k is the coefficient of kinetic friction, x_0 is the position of the vibrating core, and all other variables are as before. Essentially, the Fixed Distance model's governing equation for I_2 is used and two more equations of motion of the movable core are added. As written, the governing equation for this system are not quite correct because they do not account for the collisions. The large acceleration induced by collisions is not handled well by a finite difference method. The large coefficients make error compound correctly. MATLAB's ode15s solver, which is used to solve this model, supports event functions. They stop the integration when the it evaluates to zero. In this case, event functions are used to locate when the cores are in contact. At that point, ode15s is called again, but the initial condition for velocity is reversed. This simulates an elastic collision.

This models is an even better representation of the physics of this particular device, but it does make additional assumptions. Those assumptions are

- Collisions are elastic
- The vibrating core does so at the same frequency as the power line or harmonic thereof
- The vibrating core has a constant amplitude of vibration

Since this model requires many more physical parameters, and some like the coefficients of friction are not easy to measure, all we will concern ourselves with is the motion of the core and the shape of the output voltage, not the exact numbers.

The first test case is when there is no current in the line. In that case, the cores should not move since there is no force on them. The output for this case is shown below.

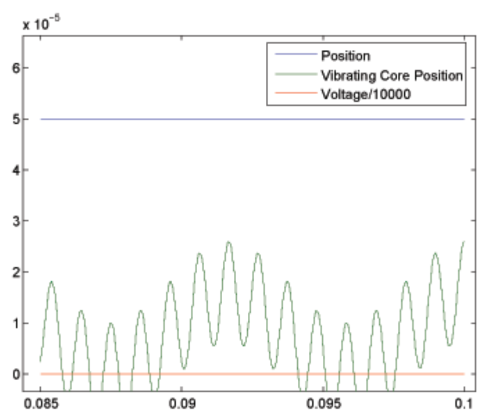

Figure 8: A plot of core separation as a function of time. The cores remain a fixed distance apart and no output voltage is predicted. Notice that the vibrating core is moving, this is not physically correct but consistent with the 3rd assumption

As expected the cores do not move and there is no output voltage. While the vibrating core did vibrate, this behavior is the result of the 3rd assumption in this model. It is not physically correct, but it does not harm this test case. The motion of the core is correct in this rather trivial case. The other test case is when the input current is constant. In this case, the c-cores are acting as permanent magnets. The predicted motion of the cores in this case is shown below.

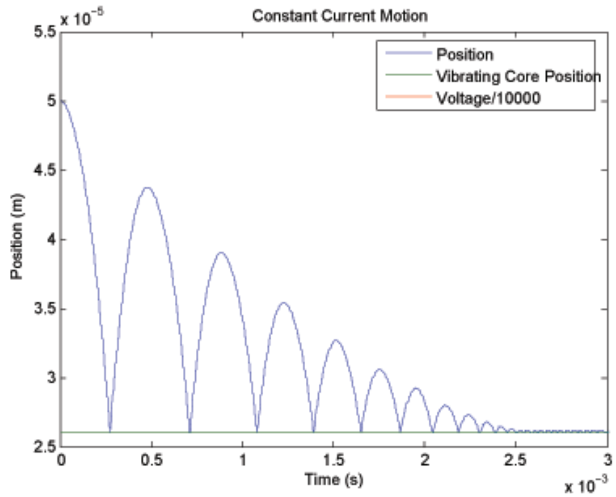

Figure 9: A plot of core separation as a function of time. The c-cores are acting as permanent magnets. As expected, they clamp together and do not release.

This behavior is expected. From childhood, we know that two magnets end to end will attract each other and not rebound after time has passed. The distance between the cores decays to zero, consistent with that behavior. It is important to note this will occur even if there is no friction. The two cores come together as a result of the magnetic force alone. From Fixed Distance model, we have verified the governing equation for output power. From the test cases of the Variable Distance model, the physical equations of motion have been verified. This model can be trusted to provide a qualitative representation of the dynamics involved in this device.

Results

The most interesting result is what is happening during normal operation of the toroid. In this case, the system is being driven by a 60 Hz sine wave. If the vibrational toroid is not allowed to move, any magnetic field is enough to hold the cores together. That would result in no air gap, which is not what we observed. The reason an air gap exists at all is because the power line's vibrations are driving the system. When the vibration of one core plus two of its harmonics are accounted for, the behavior in Figure 10 appears.

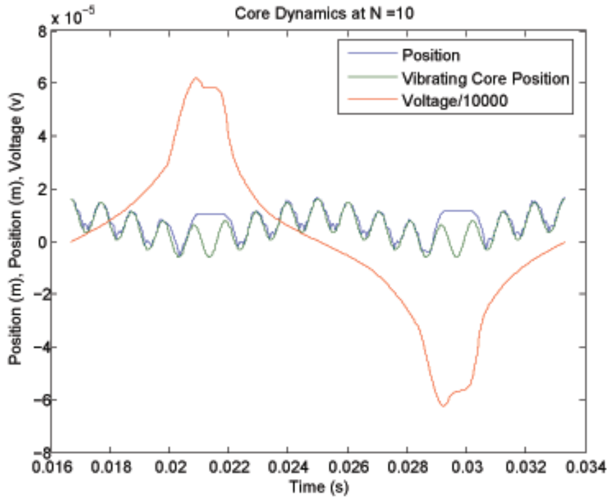

Figure 10: A plot of the output voltage waveform and the motion of the core. The cores remain together for the majority of the time. Unfortunately for the power output, the point of separation comes during the period of time the majority of the power is being drawn.

This result gives insight into when and how the cores move. For the most part, the cores stay together. That produces a 120 Hz sound, plus all the harmonics. In Figure 10, the blue and green curves, which signify the movable and vibrating core positions respectively, are within 10 microns for the majority of the time. However, at those times the cores are not outputting much voltage. The only time the cores obtain a significant distance is unfortunately the time they can output the most power. There is a trade off. When the cores extract power from a power line, they reduce the magnetic force available to hold them together. The vibrations cause them to drift apart, which reduces the power they draw. This effect creates another balancing act in the toroid. On one hand, a core with more turns is able to siphon more power, but those same cores will have less of a magnetic field to hold them together. This happens because the output voltage is related to the output current by $V = I_2R$. As V grows, the current available to counteract the magnetic field input current grows. That reduces B. This effect also depends on N as can be seen in the following figure.

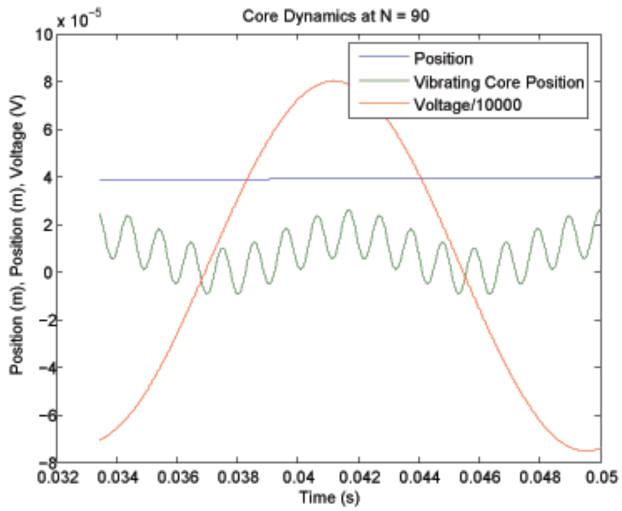

Figure 11: This is a plot of output waveform and position of the core as a function of time. With a larger value of N, the magnetic field is not able to hold the cores together. That results in them remaining a steady distance a part and producing a relatively low output voltage.

In this figure, there were $N = 90$ turns on the core. Recalling Figure 7, the theoretical voltage is too high. The reason it was too high was the core separation was too small. It was held constant from the $N = 10$ case. Comparing Figure 10 and Figure 11 it is clear why that assumption is flawed. At $N = 90$, is not enough magnetic field to hold the two cores together as the cores are trying to output too much voltage. The maximum power drawn from the core is around $N = 50$. This appears to be a good compromise when it comes to the movement of the cores too. The waveform and movement look like the following,

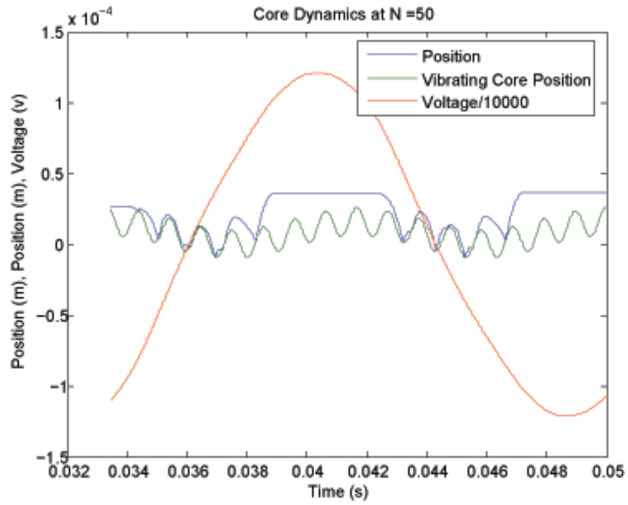

Figure 12: This is a plot of output waveform and position of the core as a function of time. It is a compromise between the $N = 10$ and $N = 90$ cases. Though the cores are further away from each other, there are more turns for them to extract power with. This combination extracts the maximum voltage from the line.

This number of turns is a compromise between drawing large amounts of voltage but being far away and drawing little power but being close. The best way to extract the maximum voltage is of course a combination of these two extremes. Although it needs calibration, this model is able to accurately reflect the dynamics of the this system.

Variable Distance Analysis

One obvious results of Figure 11 is that the core reaches a steady state. For $N > 80$, the c-cores stay approximately the same distance apart for entire periods. In these cases, it is perfectly acceptable and even beneficial to use the Fixed Distance model. It is faster and just as accurate in these cases. At smaller N and larger input currents, that model breaks down as can be seen in all the other figures. Instead, it is a better assumption to treat the cores as one unit except for the milliseconds they separate. Unfortunately, that separation occurs when the cores are in the best position to output voltage, so it is an invalid assumption to say there is no air gap.

As with the Fixed Distance model, N plays a crucial role because it affects the strength of the magnetic field. A core with a large N can output more voltage, but the weak magnetic field means it will have a larger air gap. On the other hand, if N is too small the cores will be very close. The small air gap will ensure a larger output voltage, but that core lacks the number of turns to output much voltage. Therefore, while it has access to power it cannot extract it. This explains why the compromise at $N = 50$ produced better results. It represents a compromise between access to power and the ability to draw it.

One nice benefit of this model is it displays the effects of magnetic transduction. Magnetic transduction is the changing of magnetic energy to other forms, kinetic energy in this case. While experimenting, we observed small high frequency oscillations in the output voltage. The Fixed Distance model does not display this effect, and it should not. This effect comes from motion. These tiny oscillations are displayed in the following figure.

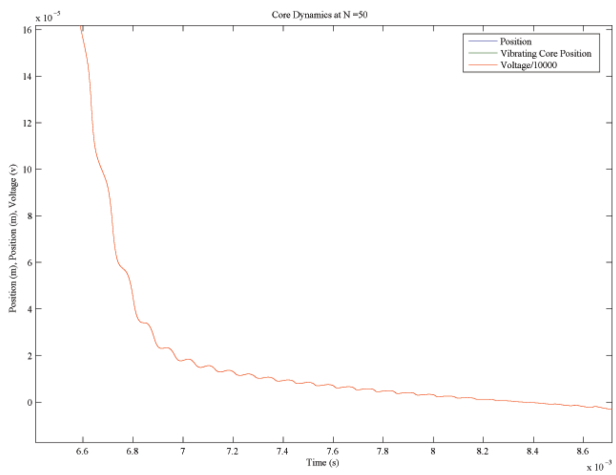

Figure 13: Effects of Transduction. Notice the low amplitude but high frequency oscillations characteristic of that effect

Judging by the size of these oscillations, they are not a reason for the failure of the original model. The image in Figure 13 is magnified several times. In addition, this effect appears when the input current is very high. It is caused when the collisions between the cores is violent enough to send them a considerable distance apart. Since this effect was observed experimentally, it is testament to the accuracy of this model that the effect shows up here. This provides a reason to be confident about the validity of these results, even though the proper parameters could not be found to allow these predictions to be compared with physical data.

Conclusion

The simple model the clinic team created was accurate within 20%, but it was 80 standard deviations away from some measured values. The magnetization and air gap are the two most important factors to consider. They reduced the maximum error to only 8 standard deviations. No other factor has a chance of that dramatic of an effect. However, even the Fixed Distance model systematically overestimates the output voltage at large N. The cause of that overestimation is the core separation's dependence on N. As N grows, the magnetic field between the cores weakens. With less force to hold them together, the average separation will grow. Past $N = 80$, the separation will even become constant. With a larger separation, a smaller voltage will be extracted. At best, this could only account for 8 standard deviations of error, but the Variable Distance model was not a waste of time. It showed that the cores are in contact most of the time, but when the output voltage rises they will separate. In addition, the model showed that the air gap itself was caused by the vibrations of the power line. That was completely unexpected. Together, these models emphasize the importance of properly selecting N. As before, N is the number of turns of wire around a c-core. While it may seem a small thing, the differences in the behavior of every figure in this paper stems from changing that number. When designing a toroidal power transformer, the most important factor, regardless of core dimensions, friction or any other force, is selecting the proper number of turns. It affects everything from the output voltage to the motion of the cores themselves.

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