The Chaotic Oscillating Magnetic Pendulum

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MATH 164 Scientific Computing

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Harvey Mudd College

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Overview

What is Chaos?

Although no formal definition of the word chaos is universally accepted, we can maintain a working definition in the following manner:

Definition

A deterministic system exhibits chaos if it:

1. exhibits long-term, aperiodic behavior, and
2. displays sensitive dependence to initial conditions.

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Introduction to the Problem

This idea of chaos is exhibited in the particular system which I chose to study, the chaotic oscillating magnetic pendulum.

The Setup

- the pendulum consists of a magnet suspended from a string
- the plane under the pendulum contains a distribution of like magnets which, based on their number and placement, affect the dynamics of the pendulum differently.

For each initial position, the trajectory of the pendulum eventually stabilizes around one of the plane magnets. However, in the interim the motion of the pendulum is chaotic, with basins of attraction for different magnets separated by fractal curves in the plane.

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Why am I really doing this?

The Beginning

When I first undertook this problem, my group used several Matlab programs to numerically analyze the system. There were a couple of problems with this:

- imprecise computing method
- Colin's question....

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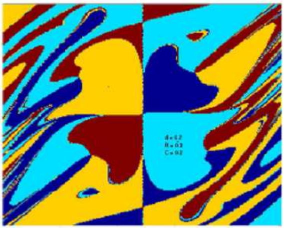
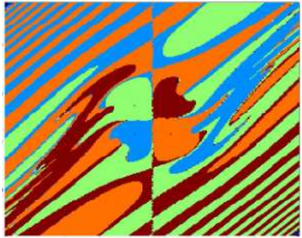
Let me tell you a little story...





Assumptions

The following assumptions are used in the derivation of the model:

- The length of the pendulum is large compared to the spacing of the magnets.
- The magnets are point attractors positioned in a plane a small distance below the pendulum.
- Magnetic forces follow an inverse squared law; i.e. the force is inversely proportional to the square of the distance.

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Parameters

- (x, y): the Cartesian coordinates of the pendulum bob
- $\{(x_i, y_i)\}$: the Cartesian coordinate of magnet i
- d: the vertical distance from the pendulum bob to the plane in which the magnets lie
- R: the friction force coefficient
- C: the gravitational (spring) force coefficient

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The derivation for the model was taken from ***Chaos and Fractals: New Frontiers of Science*** by Peitgen, Heinz-Otto; Jurgens, Hartmut; and Saupe, Dietmar, Springer-Verlag New York, Inc., 1992.

The model, however, is limited in its use because of the fractal nature of the basins of attraction (other than a few of the larger basins, it is impossible to measure accurately enough the initial position of the pendulum, to see if it coincides with theory for most positions in the plane).

Derivation of the Model

For Starters

We define the origin of the Cartesian coordinate system to be the gravitational equilibrium position of the pendulum bob, and specify coordinates of the magnets relative to this origin.

Derivation of Magnetic Forces

Distance between pendulum bob and magnet i:

$$\sqrt{(x_i - x)^2 + (y_i - y)^2 + d^2}$$

$\therefore$ magnetic force proportional to

$$\frac{1}{(x_i - x)^2 + (y_i - y)^2 + d^2}$$

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However, we must ignore the vertical component of this force, as we assume the pendulum bob to be restricted to a plane.

Magnetic Forces Continued

It can be shown that the x and y components of the magnetic force are

$$\vec{F}_x = \frac{x_i - x}{\left((x_i - x)^2 + (y_i - y)^2 + d^2\right)^{3/2}}$$

$$\vec{F}_y = \frac{y_i - y}{\left((x_i - x)^2 + (y_i - y)^2 + d^2\right)^{3/2}}$$

Other Forces

Gravity and Drag

- The gravitational force is proportional to the bob's distance away from the origin.
  - x and y components are proportional to $-x$ and $-y$ respectively
- The friction force acts in opposition to the direction of motion.
  - proportional to the velocity (x', y')

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Governing Equations!

Using Newton's Law we equate the sum of forces acting on the system to the acceleration of the mass (pendulum bob):

$$
\begin{aligned}
x'' &= -Rx' + \sum_i \frac{x_i - x}{\left((x_i - x)^2 + (y_i - y)^2 + d^2\right)^{3/2}} - Cx \\
y'' &= -Ry' + \sum_i \frac{y_i - y}{\left((x_i - x)^2 + (y_i - y)^2 + d^2\right)^{3/2}} - Cy
\end{aligned}
$$

Governing Equations!

Using Newton's Law we equate the sum of forces acting on the system to the acceleration of the mass (pendulum bob):

$$\begin{aligned} x'' + Rx' - \sum_i \frac{x_i - x}{\left((x_i - x)^2 + (y_i - y)^2 + d^2\right)^{3/2}} + Cx &= 0 \\ y'' + Ry' - \sum_i \frac{y_i - y}{\left((x_i - x)^2 + (y_i - y)^2 + d^2\right)^{3/2}} + Cy &= 0 \end{aligned}$$

Why did I choose Mathematica?

Reasons

- Symbolic and Numeric Integration
- Needed better numerical integration technique

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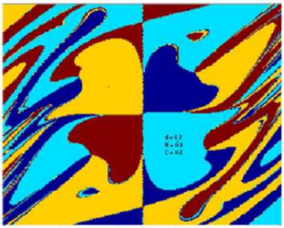
Old Matlab M-files

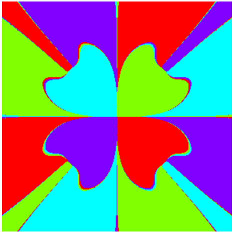
```
function pendulum(x0, y0, magx, magy, d, R, C, maxT, tol)
[T Y TE YE IE] = ode45(@odefun, [0 maxT], [x0 0 y0 0]', ...
odeset('reltol', 1e-8, ...
'abstol', 1e-9, ...
'events', @events), ...
magx, magy, d, R, C, tol);
figure;
plot(Y(:,1), Y(:,3));
axis([min(magx) - 2 max(magx) + 2 min(magy) - 2 max(magy) + 2]);

function Yprime = odefun(T, Y, magx, magy, d, R, C, tol)
x = Y(1);
xprime = Y(2);
y = Y(3);
yprime = Y(4);
Dcubed = sqrt((magx - x).^2 + (magy - y).^2 + d^2).^3;
Yprime = [xprime ; ...
-R*xprime - C*x + sum((magx - x)./Dcubed) ; ...
yprime ; ...
-R*yprime - C*x + sum((magy - y)./Dcubed)];

function [value, isterminal, direction] = ...
events(T, Y, magx, magy, d, R, C, tol)
value = zeros(size(magx));
for k = 1:length(magx)
value(k) = sqrt(sum((Y - [magx(k) 0 magy(k) 0]').^2)) >= tol;
end
isterminal = ones(size(magx));
direction = zeros(size(magx));
```

New, Improved Mathematica Notebook

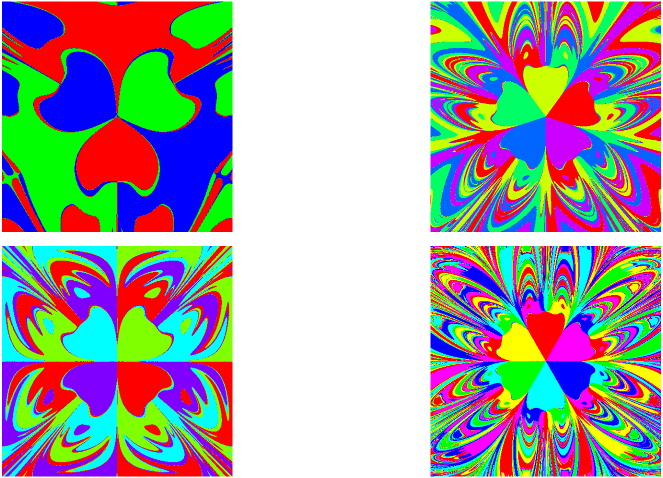
```
n = 250; d = 0.2; R = 0.15; g = 0.2;  mags = {{Sqrt[3], 1}, {-Sqrt[3], 1}, {0, -2}};
f[mag_] := (d^2 + (mag[[1]] - x[t])^2 + (mag[[2]] - y[t])^2)^1.5;
solution =
    NDSolve[{x''[t] == Plus @@ Map[(#[[1]] - x[t])/f[#] &, mags] - g x[t] - R x'[t],
             y''[t] == Plus @@ Map[(#[[2]] - y[t])/f[#] &, mags] - g y[t] - R y'[t],
             x[0] == x1, x'[0] == 0, y[0] == y1, y'[0] == 0}, {x, y}, {t, 0, 100},
             MaxSteps -> 200000];
Show[Graphics[
      RasterArray[
        Table[final = {x[100], y[100]} /. solution[[1]];
          radii = Map[(final - #).(final - #) &, mags]; r = Min[radii];
          Hue[Position[radii, r][[1, 1]]/3], {y1, -5.0, 5.0, 10.0/n},
          {x1, -5.0, 5.0, 10.0/n}]]], AspectRatio -> 1];
```

$d = .2, R = .3, g = .2$





Basins for various numbers of magnets: $d = .2, R = .2, g = .2$



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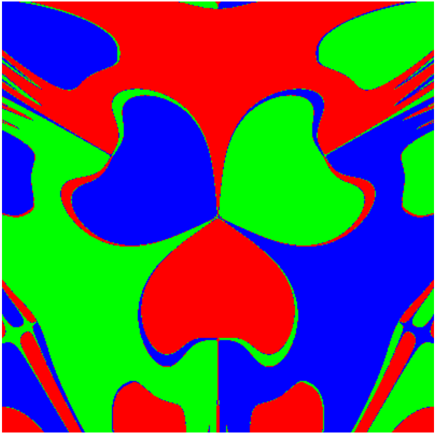

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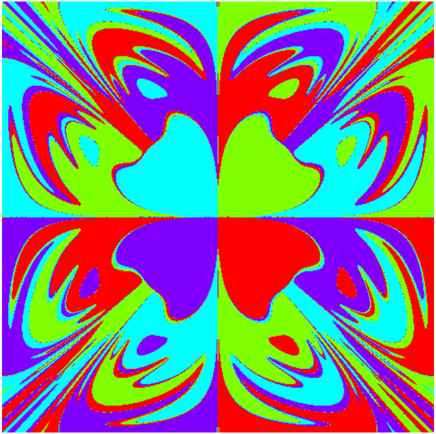

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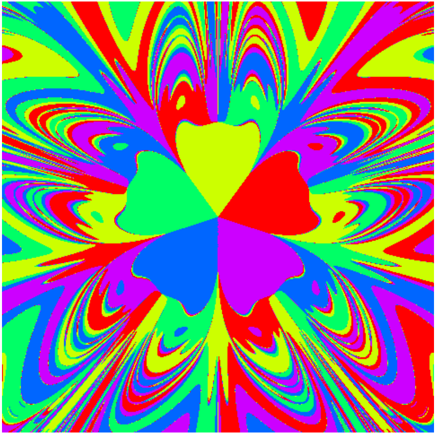

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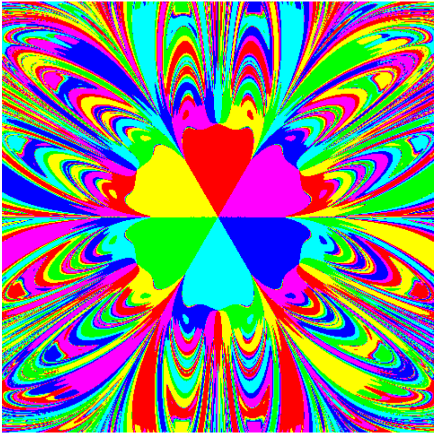
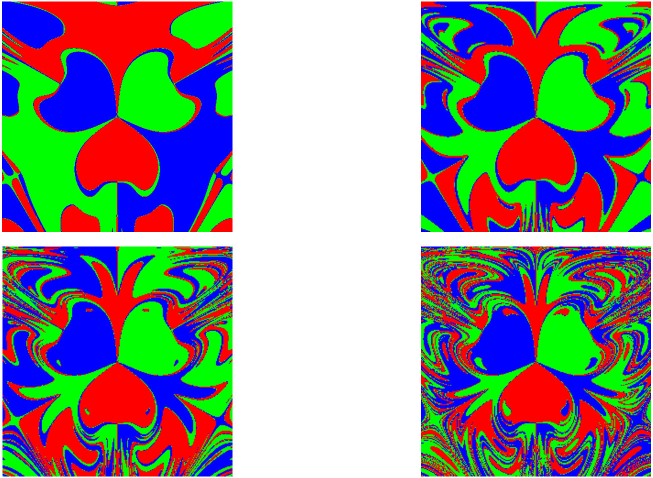
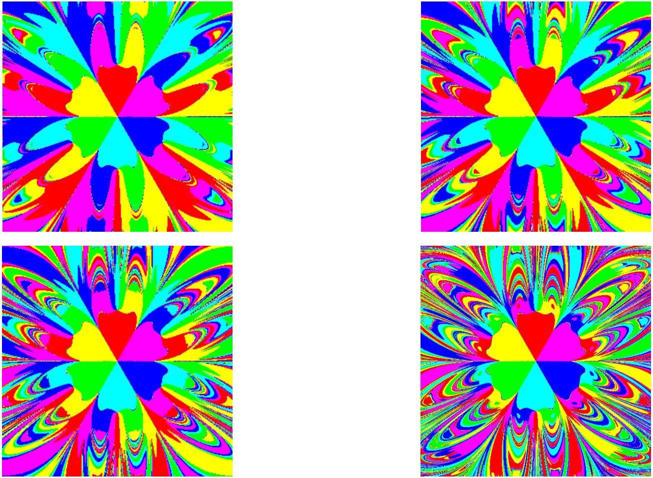


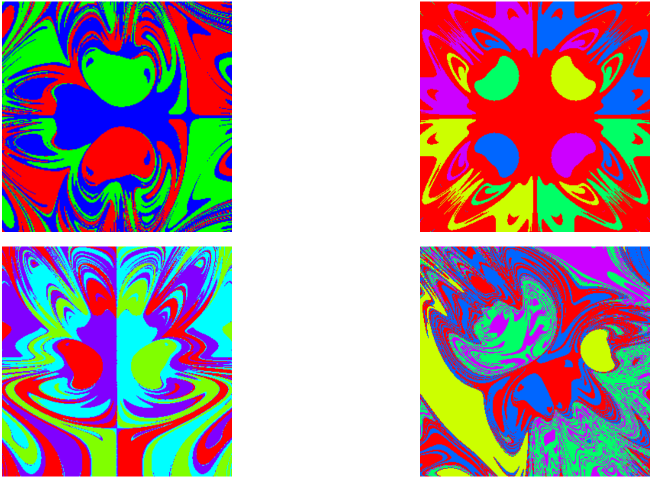
3-Basins for various values of the friction coefficient



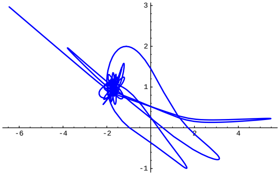
6-Basins for various values of the friction coefficient



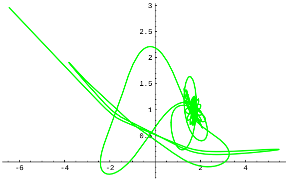
Some Unusual Basins



Some Trajectories: $d = .25$, $R = .07$, $g = .2$



$x_0 = -6.458$ and $y_0 = 2.967$



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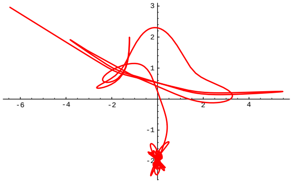

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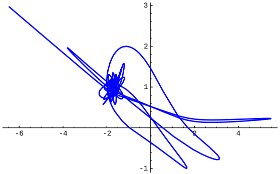

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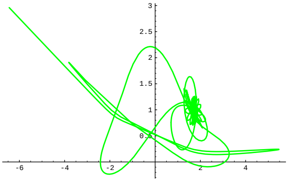

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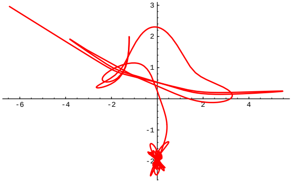

$x_0 = -6.453$ and $y_0 = 2.961$

Now here's an animation! $\Longrightarrow$ Off to Mathematica

What about the other parameters?

What have we seen?

- a little variation of the friction coefficient
- a couple wierd magnet positions

What else is there?

- gravitational coefficient?
- height of pendulum bob (d)?

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Let's investigate! $\Longrightarrow$ Back to Mathematica

Some conclusions...

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- knowing which program to use is as important as knowing how to use it

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Computing things Scientifically ROCKS THA HIZZ-OUSE!!!

References

Jürgens, Hartmut, Peitgen, Heinz-Otto, and Saupe, Dietmar, *Chaos and Fractals: New Frontiers of Science*. Springer-Verlag, New York, Inc., 1992.

http://www.sas.upenn.edu/ uak/