One-Phase Stefan Problems

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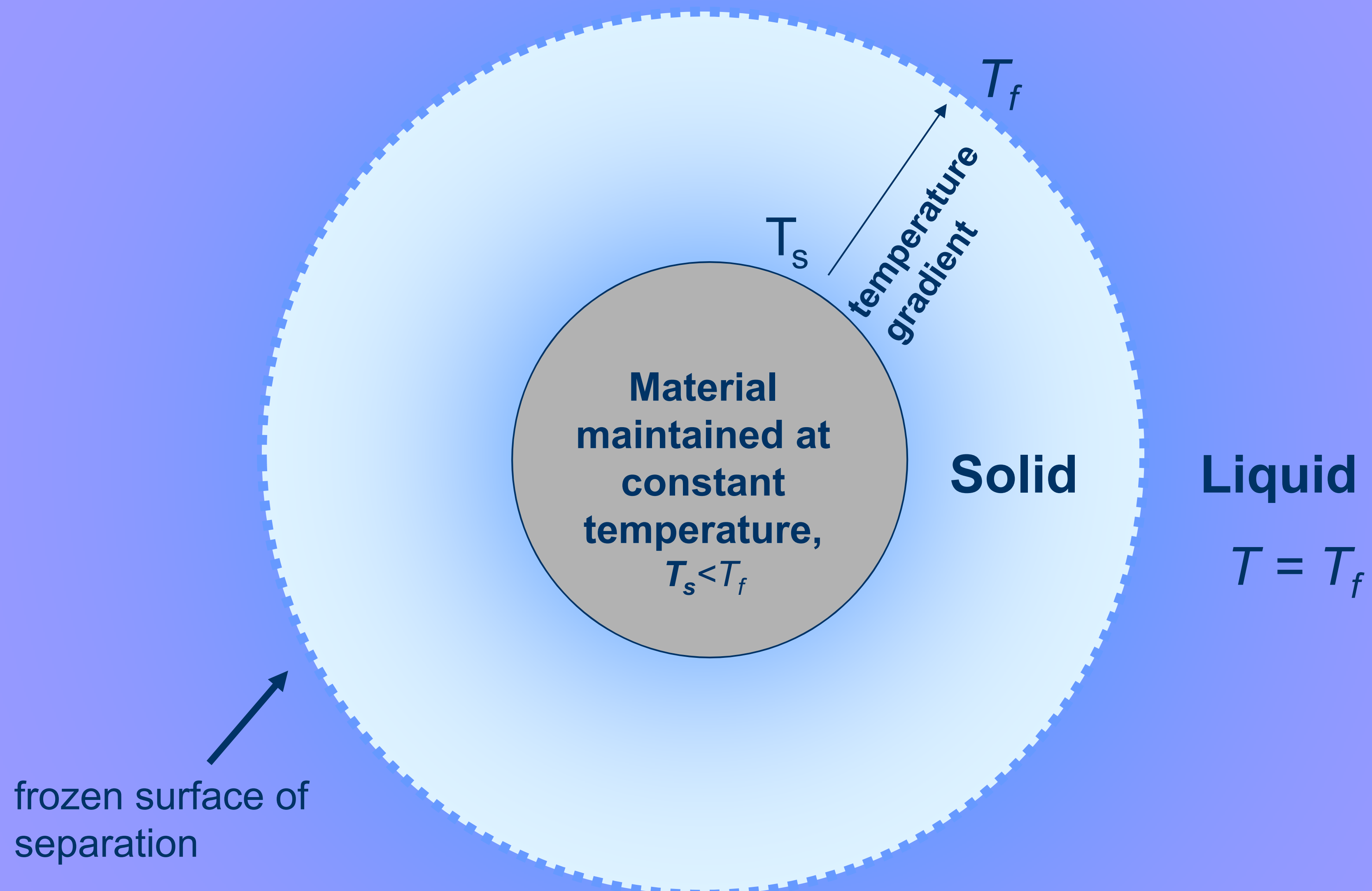
Stefan Problems

- A sub-category of moving boundary problems
- Problems involving
  - change of phase
  - moving surfaces of separation between phases

i.e. melting ice or freezing water

Idealized Solidification of a Liquid

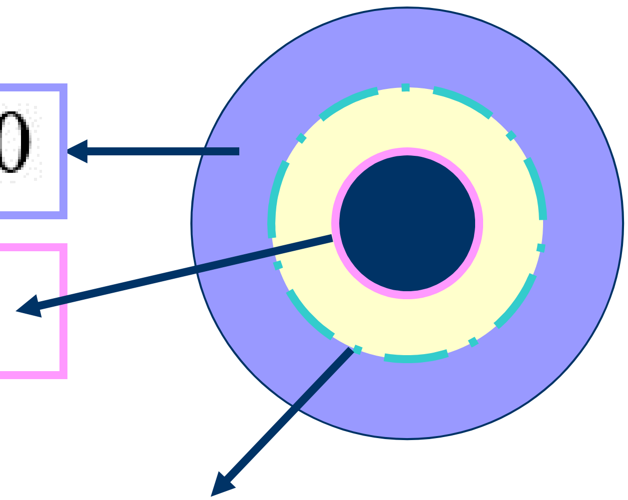


Describing the System

$$\frac{\partial T}{\partial t} = \frac{\kappa}{r^\beta}\frac{\partial}{\partial r}\left[r^\beta \frac{\partial T}{\partial r}\right], \quad a < r < R(t), \quad t > 0$$

$$T = T_\mathrm{f}, \quad r \geqslant R(t), \quad t > 0$$

$$T = T_\mathrm{s}, \quad r = a, \quad t \geqslant 0$$



Solid-liquid interface:

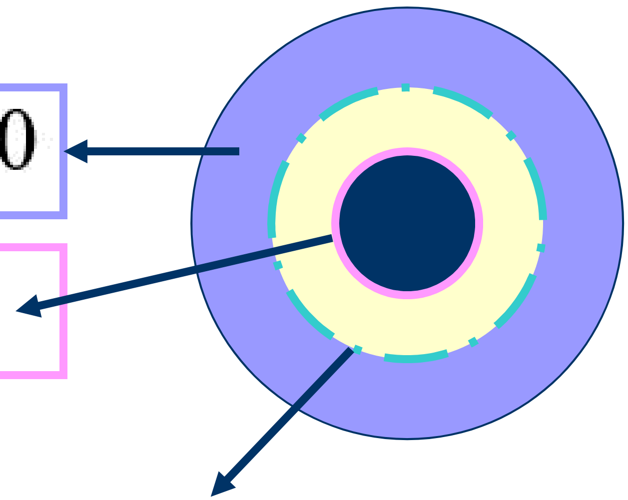
$$K\left(\frac{\partial T}{\partial r}\right)_{R(t)} = L\rho \frac{\mathrm{d}R(t)}{\mathrm{d}t}$$

Describing the System

$$\frac{\partial U}{\partial \tau} = \frac{1}{z^{\beta}} \frac{\partial}{\partial z}\left[z^{\beta} \frac{\partial U}{\partial z}\right], \quad 1 < z < Z(\tau), \quad \tau > 0$$

$$U = 1, \quad z \geqslant Z(\tau), \quad \tau > 0$$

$$U = 0, \quad z = 1, \quad \tau \geqslant 0$$



Solid-liquid interface:

$$\left(\frac{\partial U}{\partial z}\right)_{Z(\tau)} = \alpha \frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau}, \quad Z(0) = 1$$

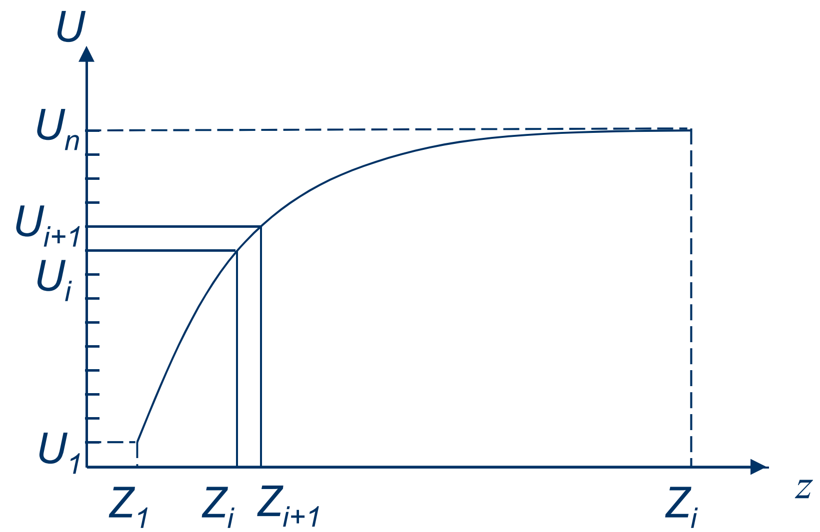
Heat Balance Integral Method

- Assume space-dependence of temperature that is consistent with the boundary conditions
- Integrate the heat flow equation and substitute assumed temperature distribution
- Solve to find the motion of the phase change boundary

HBIM

- Divide the temperature range $[0,U]$ into n equal intervals $U_1,\ldots,U_n$.

- Assume a linear profile each temperature interval



- Integrate $\int_{Z_i}^{Z_{i+1}} [z^{\beta} * (\text{heat equation})]$

HBIM

- Following this method, we can reduce the heat equation and rearrange it into a system of (nonlinear) first-order ODEs for Z_i

$$(2Z_1 + 1)\dot{Z}_1 = \frac{6}{Z_1 - 1} - \frac{6Z_1}{Z_2 - Z_1}$$

$$(2Z_{i+1} + Z_i)\dot{Z}_{i+1} + (Z_{i+1} + 2Z_i)\dot{Z}_i = \frac{6Z_i}{Z_{i+1} - Z_i} - \frac{6Z_{i+1}}{Z_{i+2} - Z_{i+1}}, \quad i = 1, 2, \ldots, n-2$$

$$[2(1 + 3\alpha n)Z_n + Z_{n-1}]\dot{Z}_n + (Z_n + 2Z_{n-1})\dot{Z}_{n-1} = \frac{6Z_{n-1}}{Z_n - Z_{n-1}}$$

Finite Difference

- Runge-Kutta method is well-suited; however,
  - it requires 4 function calls each step
  - it requires a very small time step to remain stable
- Finite Difference is also a candidate
  - it requires 1 function call each step
  - it gives similar level of accuracy

Current Method

- Euler's Method:
  - Solve for $\dot{Z}(\tau_0)$
  - $\dot{Z}(\tau_i) = Z(\tau_{i-1}) + \dot{Z}(\tau_i) * \Delta\tau$
- How to choose τ_0 ?

Small Time Series

$$Z_i(\tau) = 1 + \mu_{i,0}\tau^{1/2} + \mu_{i,1}\tau + \mu_{i,2}\tau^{3/2} \ldots$$

- Solve for coefficients $\mu_{i,j}$ by plugging the series into our system of differential equations

Small Time Series

- n = 1, works out neatly
- n = 2, 3, …, involves non-linear systems of equations

For the coefficients to each of the three terms, rewrite system of equations as

$$\boldsymbol{F}(\boldsymbol{\mu}) = 0$$

We can then solve the system using the relation

$$\boldsymbol{J}(\boldsymbol{\mu}^{(k)}) * (\boldsymbol{\mu}^{(k+1)} - \boldsymbol{\mu}^{(k)}) = -\boldsymbol{F}(\boldsymbol{\mu}^{(k)})$$

$$\mu_{1,0} = \frac{4}{\mu_{1,0}} - \frac{4}{\mu_{2,0} - \mu_{1,0}}$$

$$\mu_{1,0}^2 + 3\mu_{1,1} = 6\left[\frac{\mu_{2,1} - \mu_{1,1}}{(\mu_{2,0} - \mu_{1,0})^2} - \frac{\mu_{1,0}}{\mu_{2,0} - \mu_{1,0}} - \frac{\mu_{1,1}}{\mu_{1,0}^2}\right]$$

$$2\mu_{1,0}\mu_{1,1} + 3\mu_{1,2} = 4\left[\frac{\mu_{2,2} - \mu_{1,2}}{(\mu_{2,0} - \mu_{1,0})^2} + \frac{\mu_{1,0}(\mu_{2,1} - \mu_{1,1})}{(\mu_{2,0} - \mu_{1,0})^2} - \frac{(\mu_{2,1} - \mu_{1,1})^2}{(\mu_{2,0} - \mu_{1,0})^3} - \frac{\mu_{1,1}}{\mu_{2,0} - \mu_{1,0}} + \frac{\mu_{1,1}^2}{\mu_{1,0}^3} - \frac{\mu_{1,2}}{\mu_{1,0}^2}\right]$$

$$\mu_{i+1,0} + \mu_{i,0} = \frac{4}{\mu_{i+1,0} - \mu_{i,0}} - \frac{4}{\mu_{i+2,0} - \mu_{i+1,0}}$$

$$6(\mu_{i+1,1} + \mu_{i,1}) + \mu_{i+1,0}(2\mu_{i+1,0} + \mu_{i,0}) + \mu_{i,0}(\mu_{i+1,0} + 2\mu_{i,0}) = 12\left[\frac{\mu_{i,0}}{\mu_{i+1,0} - \mu_{i,0}} - \frac{\mu_{i+1,1} - \mu_{i,1}}{(\mu_{i+1,0} - \mu_{i,0})^2} - \frac{\mu_{i+1,0}}{\mu_{i+2,0} - \mu_{i+1,0}} + \frac{\mu_{i+2,1} - \mu_{i+1,1}}{(\mu_{i+2,0} - \mu_{i+1,0})^2}\right]$$

$$\begin{aligned} &9(\mu_{i+1,2} + \mu_{i,2}) + \mu_{i+1,0}(2\mu_{i+1,1} + \mu_{i,1}) + \mu_{i,0}(\mu_{i+1,1} + 2\mu_{i,1}) + 2\mu_{i+1,1}(2\mu_{i+1,0} + \mu_{i,0}) \\ &+ 2\mu_{i,1}(\mu_{i+1,0} + 2\mu_{i,0}) = 12\left[\frac{\mu_{i,1}}{\mu_{i+1,0} - \mu_{i,0}} - \frac{\mu_{i,0}(\mu_{i+1,1} - \mu_{i,1})}{(\mu_{i+1,0} - \mu_{i,0})^2}\right. \\ &+ \frac{(\mu_{i+1,1} - \mu_{i,1})^2}{(\mu_{i+1,0} - \mu_{i,0})^3} - \frac{\mu_{i+1,2} - \mu_{i,2}}{(\mu_{i+1,0} - \mu_{i,0})^2} - \frac{\mu_{i+1,1}}{\mu_{i+2,0} - \mu_{i+1,0}} \\ &\left. + \frac{\mu_{i+1,0}(\mu_{i+2,1} - \mu_{i+1,1})}{(\mu_{i+2,0} - \mu_{i+1,0})^2} - \frac{(\mu_{i+2,1} - \mu_{i+1,1})^2}{(\mu_{i+2,0} - \mu_{i+1,0})^3} + \frac{\mu_{i+2,2} - \mu_{i+1,2}}{(\mu_{i+2,0} - \mu_{i+1,0})^2}\right], \quad i = 1, 2, 3, \ldots, n-2 \end{aligned} \tag{13}$$

$$(1 + 2\alpha n)\mu_{n,0} + \mu_{n-1,0} = \frac{4}{\mu_{n,0} - \mu_{n-1,0}} \tag{14}$$

$$\begin{aligned} &(6 + 12\alpha n)\mu_{n,1} + 6\mu_{n-1,1} + \mu_{n,0}[(2 + 6\alpha n)\mu_{n,0} + \mu_{n-1,0}] + \mu_{n-1,0}(\mu_{n,0} + 2\mu_{n-1,0}) \\ &= 12\left[\frac{\mu_{n,0}}{\mu_{n,0} - \mu_{n-1,0}} - \frac{\mu_{n,1} - \mu_{n-1,1}}{(\mu_{n,0} - \mu_{n-1,0})^2}\right] \end{aligned} \tag{15}$$

$$\begin{aligned} &(9 + 18\alpha n)\mu_{n,2} + 9\mu_{n-1,2} + 2\mu_{n,1}[(2 + 6\alpha n)\mu_{n,0} + \mu_{n-1,0}] + 2\mu_{n-1,1}(\mu_{n,0} + 2\mu_{n-1,0}) \\ &+ \mu_{n,0}[(2 + 6\alpha n)\mu_{n,1} + \mu_{n-1,1}] + \mu_{n-1,0}(\mu_{n,1} + 2\mu_{n-1,1}) \\ &= 12\left[\frac{\mu_{n-1,1}}{\mu_{n,0} - \mu_{n-1,0}} - \frac{\mu_{n-1,0}(\mu_{n,1} - \mu_{n-1,1})}{(\mu_{n,0} - \mu_{n-1,0})^2} + \frac{(\mu_{n,1} - \mu_{n-1,1})^2}{(\mu_{n,0} - \mu_{n-1,0})^3} - \frac{\mu_{n,2} - \mu_{n-1,2}}{(\mu_{n,0} - \mu_{n-1,0})^2}\right] \end{aligned} \tag{16}$$

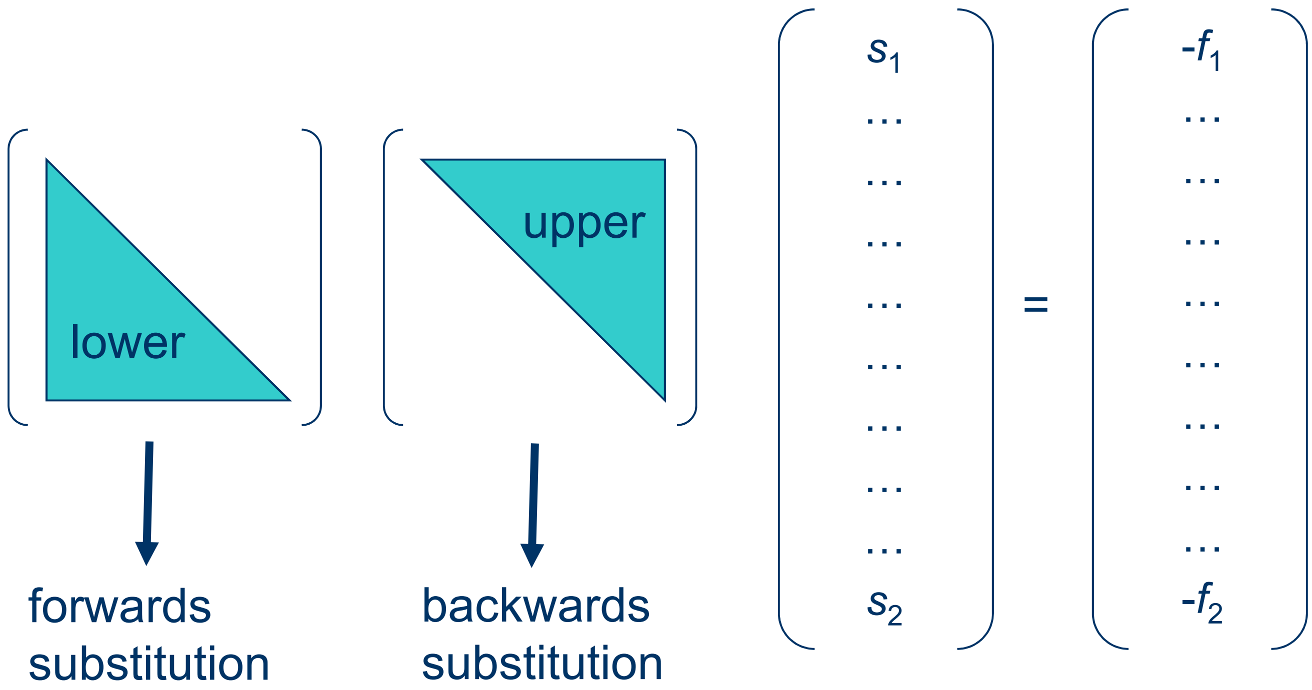
Small Time Series

$$J(\mu^{(k)}) \quad * \quad s \quad = \quad -F(\mu^{(k)})$$

$$\begin{pmatrix} a_1 & b_1 & 0 & \cdots & \cdots & \cdots & 0 \\ c_2 & a_2 & b_2 & 0 & \cdots & \cdots & \cdots \\ 0 & C_3 & \ddots & \ddots & 0 & \cdots & \cdots \\ \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ \cdots & \cdots & 0 & \ddots & a_{n-2} & b_{n-2} & 0 \\ \cdots & \cdots & \cdots & 0 & c_{n-1} & a_{n-1} & b_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & c_n & a_n \end{pmatrix} \begin{pmatrix} s_1 \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ s_2 \end{pmatrix} = \begin{pmatrix} -f_1 \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ -f_2 \end{pmatrix}$$

Small Time Series

$$J(\mu^{(k)}) \quad * \quad s \quad = \quad -F(\mu^{(k)})$$



What's Next?

- Fourth-order Runge-Kutta Method

- Tackle the spherical case, where $\beta = 2$.