Modeling Wave Propagation Using Shallow Water Equation

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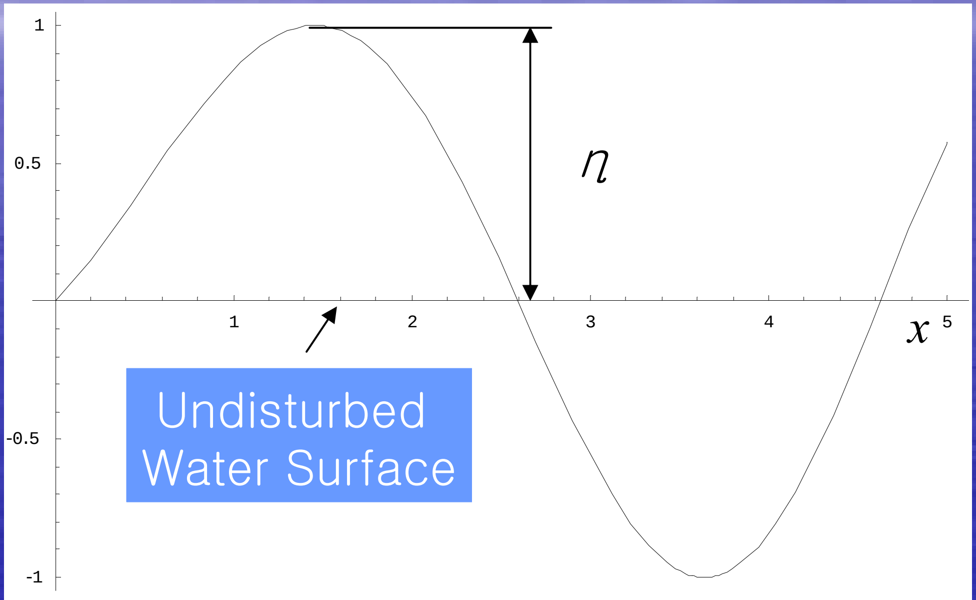
Characteristic of Shallow Water Eq.

- No pressure variance in vertical direction → Long wavelength
- Mass/Momentum Conservation
- Inviscid fluid flow

Baby Shallow Water Eq. (1D)

$$\frac{\partial \eta}{\partial t} + \frac{\partial M}{\partial x} = 0$$

$$\frac{\partial M}{\partial t} + gD\frac{\partial \eta}{\partial x} = 0$$



η : water level relative to equilibrium level

D : Total thickness of water

M : depth averaged velocity in $+x$ direction* D

Numerical Scheme (I)

$$\frac{\partial \eta}{\partial t}+\frac{\partial M}{\partial x}=0$$

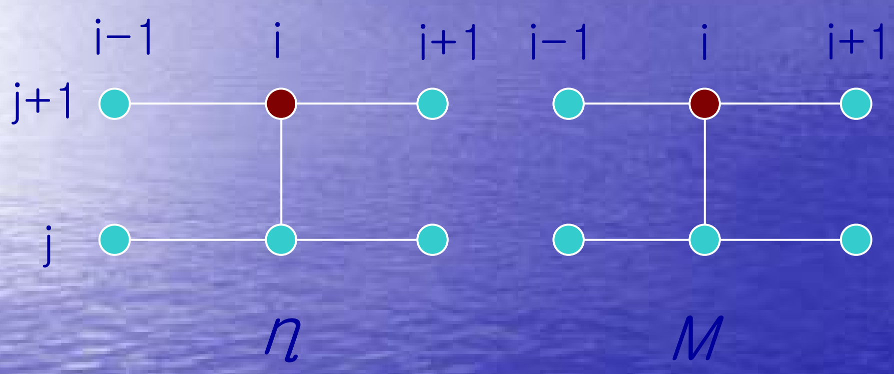
$$\frac{\partial M}{\partial t}+gD\frac{\partial \eta}{\partial x}=0$$

- Finite Difference Method
  - 2^{nd} order centered difference : $\partial/\partial x$
  - 2^{nd} order centered difference : $\partial/\partial t$

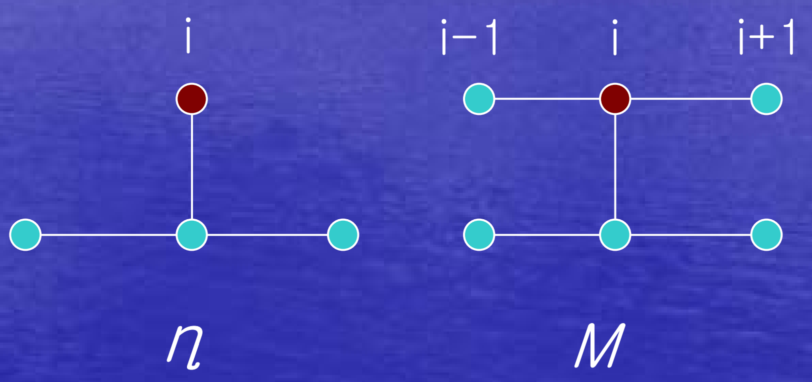
    (Crank–Nicolson Method)
- Boundary condition
  - $M = 0$ for boundaries (no velocity)

Numerical Scheme (II)
Implicit vs Explicit

Implicit



Semi-Explicit



$$\frac{\partial \eta}{\partial t} + \frac{\partial M}{\partial x} = 0$$

$$\frac{\partial M}{\partial t} + gD\,\frac{\partial \eta}{\partial x} = 0$$

→ Finite Difference Method

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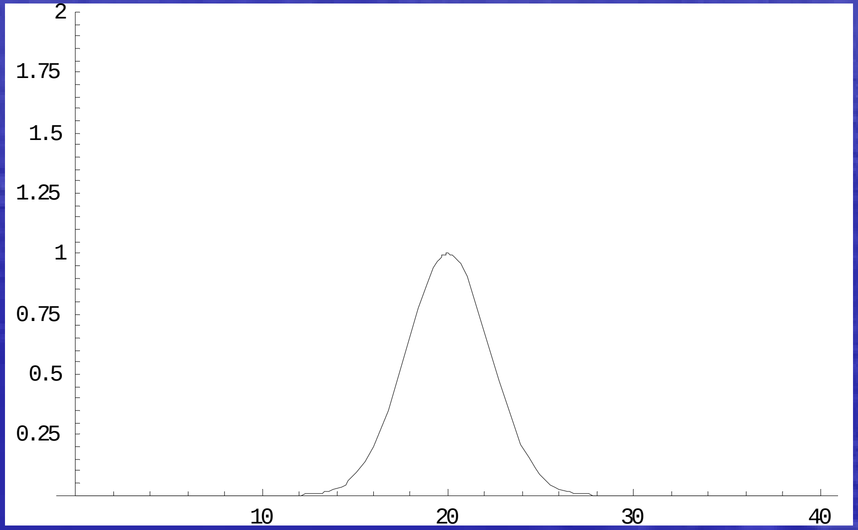
Three Initial Conditions

Sine curve

Application: Tsunami wave approaching shore

- Has long wavelength (~100 km)
- small amplitude (~1m)
- Travels at 430 mi/hr

- Initial Condition
  - η
  - M



A Gaussian

Uncoupled SWEq.

$$\frac{\partial^2 M}{\partial t^2} = g(\eta + h)\frac{\partial^2 M}{\partial x^2} - \frac{1}{\eta + h}\frac{\partial^2 M}{\partial t \partial x}$$

$$\frac{\partial^2 \eta}{\partial t^2} = g(\eta + h)\frac{\partial^2 \eta}{\partial x^2} + g\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial h}{\partial x}\frac{\partial \eta}{\partial x}$$

→ Speed of Wave Propagation

$= \sqrt{g(\eta + h)} = \sqrt{gD}$

$D = \eta + h$ (total thickness of water)

h : depth profile

Big Shallow Water Eq. (2D)

$$\frac{\partial \eta}{\partial t} + \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$$

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial x}\left(\frac{M^2}{D} + \frac{1}{2} g \eta^2\right) + \frac{\partial}{\partial y}\left(\frac{MN}{D}\right) + gh\frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x}\left(\frac{MN}{D}\right) + \frac{\partial}{\partial y}\left(\frac{N^2}{D} + \frac{1}{2} g \eta^2\right) + gh\frac{\partial \eta}{\partial y} = 0$$

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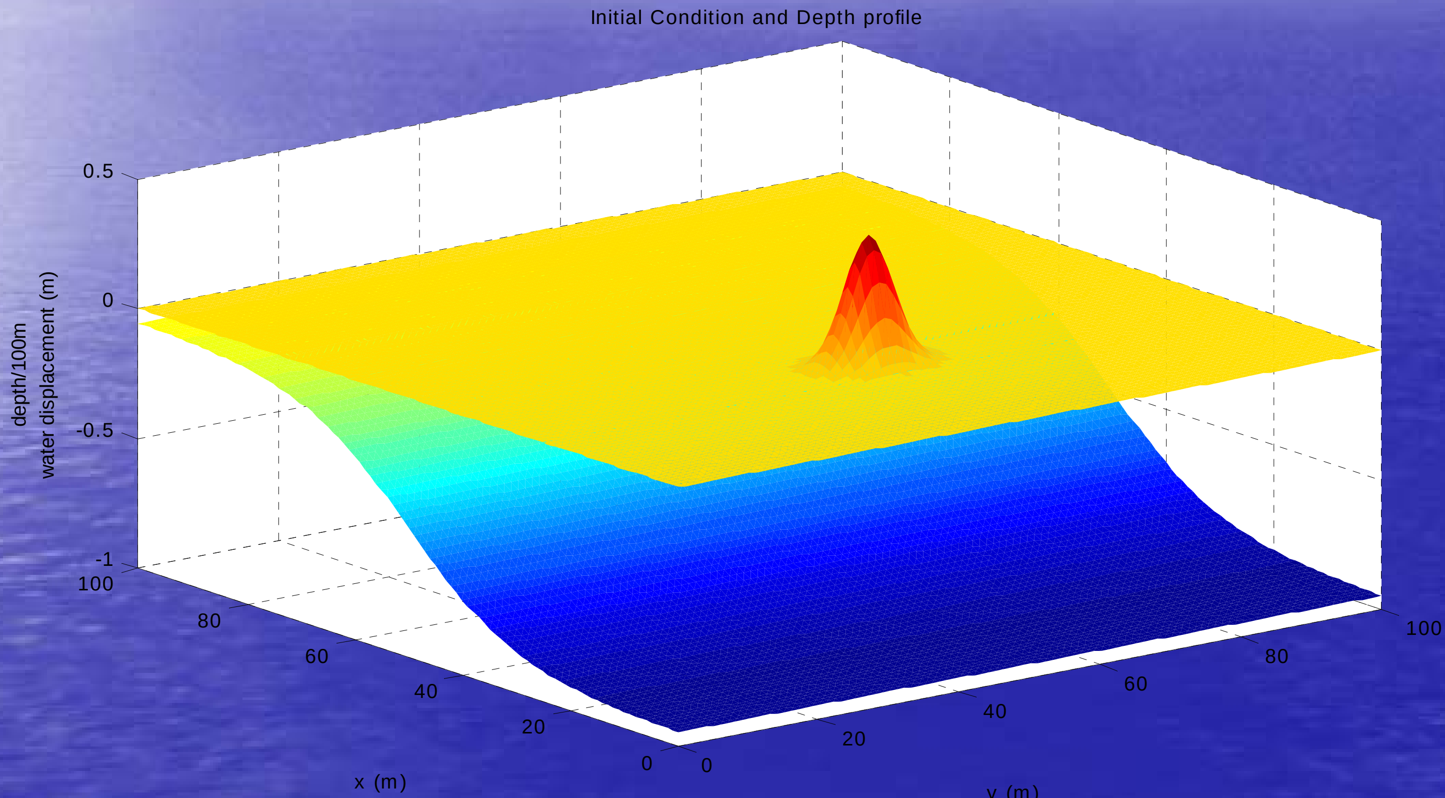
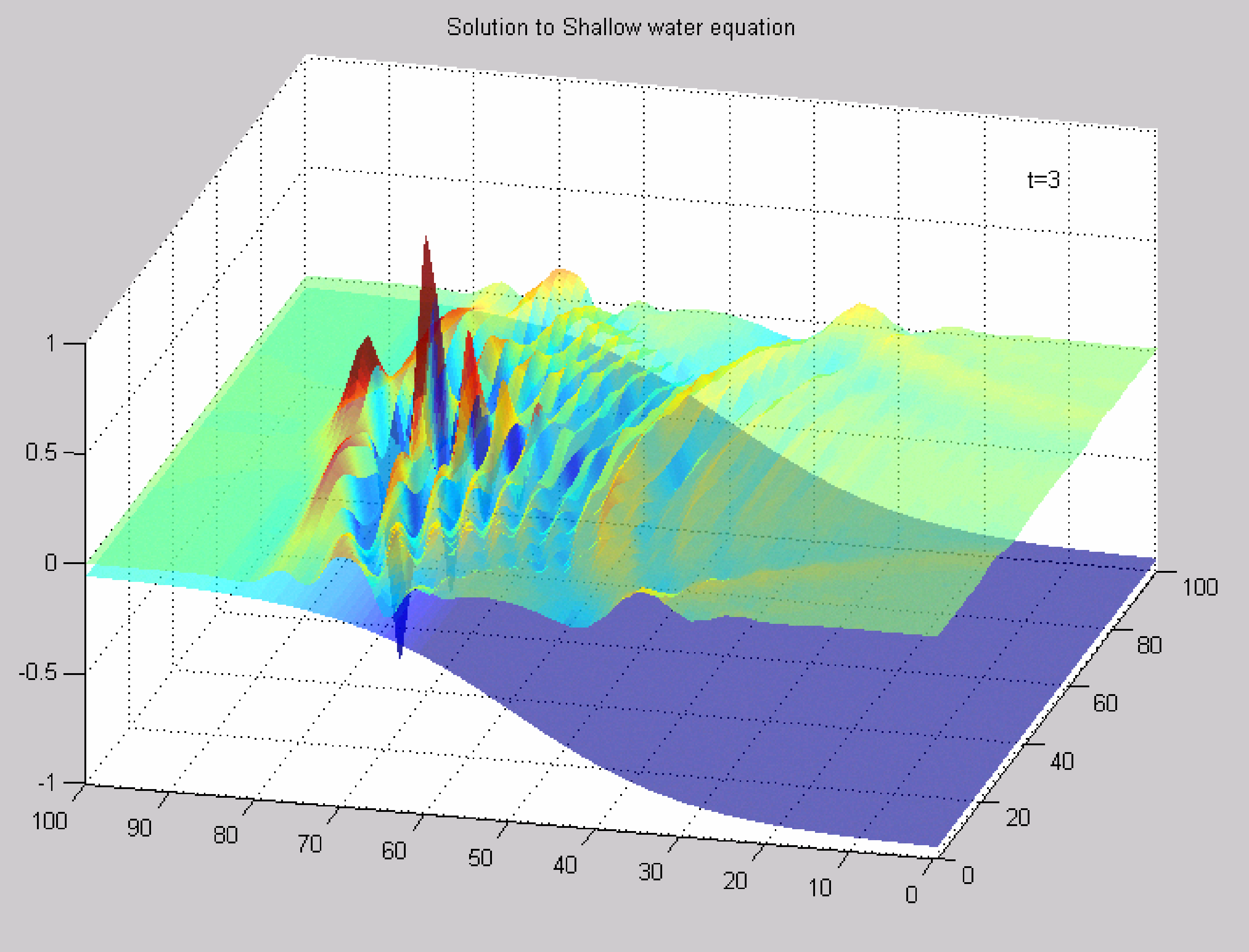
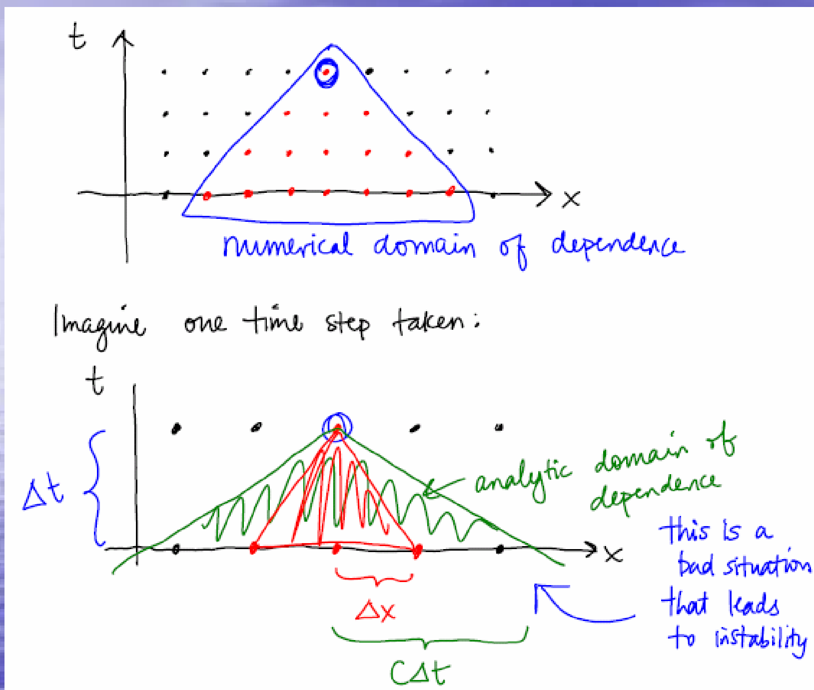
Simulation: Initial Condition / Depth profile

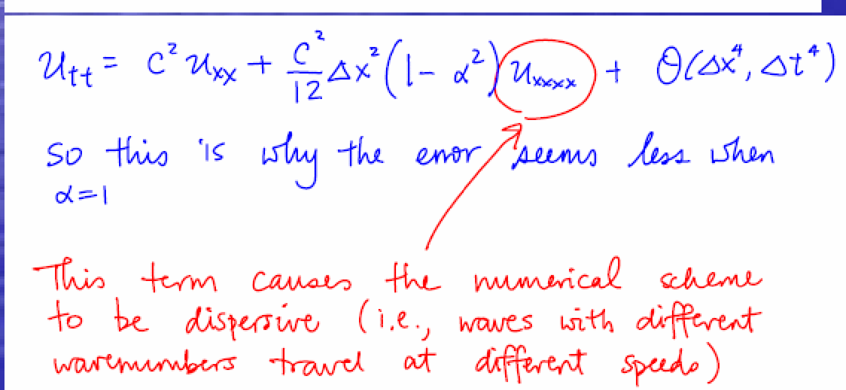


Numerical Stability



The Key

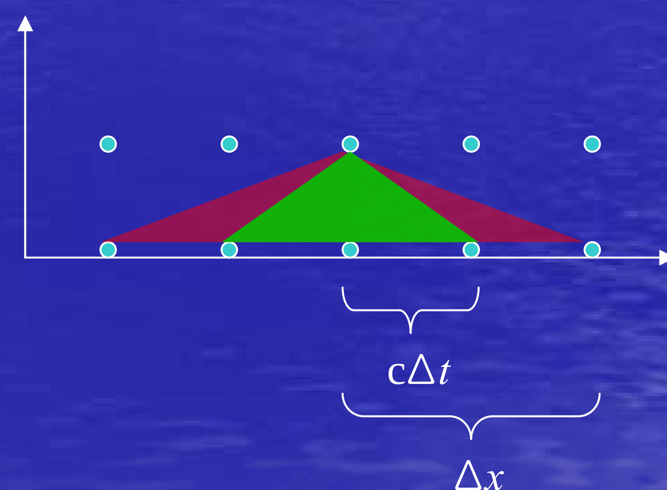




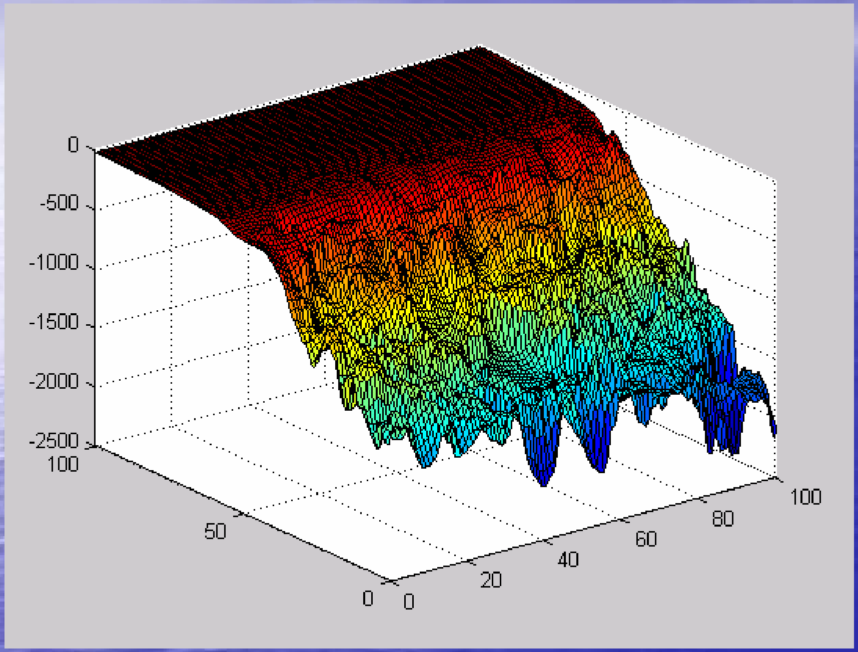
- Analytic domain of dependence is determined by the speed of wave propagation
- Speed of Wave Propagation

$$c = \sqrt{g(\eta + h)} = \sqrt{gD}$$

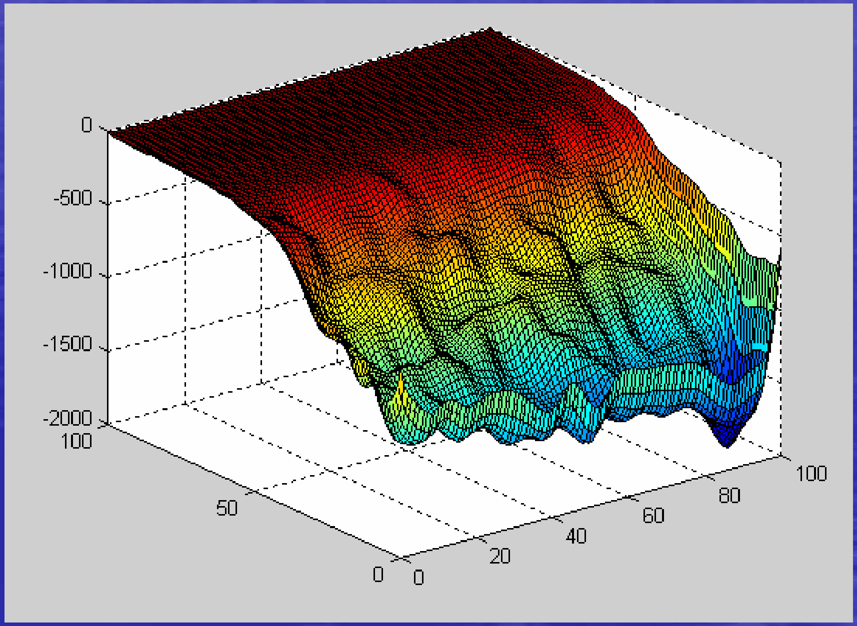
- As $D\downarrow$, $c\downarrow$



Gulf of Mexico



convolution



Take the convolution of each point with a Gaussian

Further Work

- Adaptive grid $\rightarrow$ $\Delta x\,(h)$

This is would account for speed variation

- Effective Equation Analysis
- Flux Limiter

Acknowledgement

- Professor Yong
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