# The Put Option:
# American, Bermudan, and European

Wyatt Toolson
Harvey Mudd College

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## 1 Introduction

Options, the right to buy or sell a stock at a certain price regardless of the actual value, have provided investors with a highly leveraged, high risk market for many years. People often hedge their portfolios with options as a means to further increase yields or limit losses.

There are two types of options. The first is a call option, which gives the holder the right to buy a stock for at the strike price, where the strike price is the amount the stock will be sold for. Thus if you were to own the 85 call for IBM, you would only have to pay $85 even though the stock is currently trading at $95. A put option gives the holder the right to sell the stock at the strike price. If you had the 85 put for IBM and the stock plummeted to $70, you would still be paid $85 per share.

However, options are not indefinite contracts and have exercise dates associated with them. You can often find options for stocks with exercise dates for the next several months, and oftentimes as far as a year or two away. This gives the investor many different choices to make in her investment strategy.

An option can also be either "in-the-money" or "out-of-the-money", depending on whether the option is a call or put and if the strike price is above or below the current market value of the stock. A call is said to be in-the-money if the strike price is below the stock price, and out-of-the-money if it is above. The terms are opposite for puts. Clearly an in-the-money option should be worth, or cost, more than an out-of-the-money option because it would not make sense to exercise a 100 call for IBM when you could buy it on the market for $95.

We can also refer to "American", "Bermudan", or "European" options (there are many other choices but these are the three most common). As mentioned, an option has an expiration date where it is either exercised, and the stock changes hands, or it expires worthless because the stock never reached a price where the option was in-the-money. A European option is one where the holder is only allowed to exercise the option on the expiration date. On the other hand, an American option allows the holder to exercise it whenever she would like as long as it is in-the-money. A Bermudan option falls somewhere in between as it usually has one day per month that it can be exercised early as opposed to the final day.

This concept introduces us to the premium on the price of the option. The premium is simply an amount of money that the person selling the option tacks onto the value in order to compensate them for the risk involved. Again, thinking logically, it does not make sense that a person would sell an option for the same value of the stock. In the IBM case, an 85 call would not be worth merely the $10 between the stock and strike price, but, since it is in-the-money, it would have a premium tacked on to take it to $10.50. This additional $0.50 is the only motivation that the person selling the option has, otherwise they would just hold on to the stock themselves.

But how much should this premium be? We can establish some general guidelines that help us to determine how much it should cost.

- **The longer to the expiration date, the higher the premium** This makes sense, because a longer amount of time would make it more likely that the option would become in-the-money.

- **The farther from at-the-money an option is, the lower the premium** As you get deeper in, or out of, the money, you approach an option that is either equivalent to a share of stock itself or that has no chance of ever being valuable, and thus it should be priced as such.

- **The higher the volatility of the stock, the higher the premium** Volatility is a measure of how much a stock's price can change, so the higher it is means that it could easily jump above the strike price and become valuable.

- **The higher the risk-free yield, the lower the premium** As the risk-free return increases, the opportunity cost of buying the option rises. The premium must drop otherwise there would be no market as buyers would rather have a guaranteed return than a risky one.

While these help us understand general trends in the final price, they are by no means an absolute. So how exactly should we determine what the price of an option should be?

## 2 The Black-Scholes Model

In a 1973 paper titled *The pricing of options and corporate liabilities*, Fischer Black and Myron Scholes described a PDE that determined the value of an option. Beginning with Itô's Lemma, a formula for the differential of a function for a stochastic process, they were able to modify it to the financial world and create a PDE that could accurately price stock options.

Let us first outline some assumptions that are inherent in the derivation and performance of the Black-Scholes PDE that we will be working with:

- **The stock price follows a lognormal random walk** This implies that the change in price of a stock will follow a lognormal probability distribution.
- **The risk-free rate of return $r$ and stock volatility $\sigma$ are known and constant** Here $r$ is the guaranteed return from something like a bank account, and $\sigma$ measures the standard deviation of the returns from the stock.
- **The stock pays no dividends** Many stocks pay dividends, which is a direct payment to the holder in terms of a percentage of the amount of stock owned.
- **There are no opportunities for arbitrage** This statement means that any opportunity for risk-free profit above $r$ will be instantly corrected by a shift in prices.
- **Trading of stocks is constant and liquid** There are no restrictions to when you can buy or sell a stock and there is always a market. This allows us to make later assumptions on the continuity of functions.

Black and Scholes were able to take these assumptions, as well as the behavior of the premiums that were on the market, and transform them to a PDE. Using Itô's Lemma, which governs the development of a certain form of stochastic PDEs, they were able to develop the famous Black-Scholes equation,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1}$$

For a full derivation, see [1] or [2].

## 2.1 The European Put Option

Let us define the value of a European put option to be $P(S,t)$. Note that the Black-Scholes equation was derived for all options, so we merely replace the value of any option $V(S,t)$ with $P(S,t)$. Let $T$ be the expiration date of the option. As mentioned before, a European option may be exercised on this day alone. In this case we have a well-defined PDE with absolute boundary conditions.

On the expiration date, $t = T$ the final payoff is

$$P(S,T) = \max(E - S, 0) \tag{2}$$

where $E$ is the exercise price. This gives us a final boundary condition. Now let us assume that if $S \to 0$, then it will be stuck there, as the only time this could happen is if the company disassembles or is delisted. This provides us with another boundary condition of

$$P(0,t) = Ee^{-r(T-t)}. \tag{3}$$

This is a payoff that is slightly discounted by the amount of time remaining until the expiration date, $T-t$ and the risk-free rate of return $r$. Finally, if the value of the stock increases without bound our chances of being able to exercise our put for a profit decrease, giving us

$$P(S,t) \to 0 \quad \text{as} \quad S \to \infty. \tag{4}$$

With these known boundary conditions, we can actually find a closed form solution for the European option [**?**]. The solution takes on the form

$$P(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1) \tag{5}$$

where $N(\cdot)$ is the cumulative distribution function for the normal distribution and

$$d_1 = \frac{\log(S/E) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\log(S/E) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

## 2.2 Transformation of Variables

Through a simple transformation of variables [2], the Black-Scholes equation can be transformed into the diffusion equation. While we are not necessarily concerned with the end result, the transformation of variables ends up being very useful when we look to solve the free-boundary version of Black-Scholes for an American option. Recall the European put option:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0. \tag{6}$$

with

$$P(S,T) = \max(E-S,0), \qquad P(0,t) = Ee^{-r(T-t)}$$

and

$$P(S,t) \to 0 \quad \text{as} \quad S \to \infty.$$

Let us now consider the change of variables,

$$S = Ee^x, \quad t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, \quad P = Ev(x,t). \tag{7}$$

This changes the Black-Scholes equation to

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv$$

where $k = \frac{r}{\frac{1}{2}\sigma^2}$. This modifies the initial condition to be

$$v(x,0) = \max(e^x - 1, 0).$$

We see that $k$ is a dimensionless constant, and it along with the dimensionless time to maturity $\frac{1}{2}\sigma^2 T$ are the only two independent parameters in the model.

If we now let

$$v = e^{\alpha x + \beta\tau} u(x, \tau)$$

we find that

$$\alpha = \frac{1}{2}(k-1), \quad \beta = -\frac{1}{4}(k+1)^2.$$

With only a slight continuation of the algebra, we would arrive at the diffusion equation. However, this is far enough for our purposes.

# 3 The American Put Option

As mentioned, American options differ in that the holder can exercise at any point before the maturity date. The effect of this right is to change the strict equality found in (2) to

$$P(S, t) \geq \max(E - S, 0). \tag{8}$$

The reason behind this falls under our assumption that there are no arbitrage opportunities. If $P$ was able to fall below $\max(E - S, 0)$, you could make a profit by buying the stock at $S$ and the option at $P$ and exercise the option at $E$. This would lead to a risk-free profit of $E - P - S$.

The difference between an American option and an European option lies when $P(S, t) > \max(E - S, 0)$. This is the point at which it is more profitable to exercise the option early, at some time $t$. We also see that herein lies the more complicated nature of the American option, as we are left to not only determine the value, but also if it should be exercised for each value of $S$. A further point can be made that for the simpler case of the European option, a closed form solution has been found and well documented. We can bypass all numerical approximations and simply plug and chug our way to a solution. However, at present time, no closed form solution for the American option exists (a solution has claimed to have been found by Song-Ping Zhu).

This same argument now causes a slight modification to the Black-Scholes equation (6) itself. Rather than being a strict equality, we now have another inequality. It now becomes,

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \leq 0. \tag{9}$$

Let us denote $S_f(t)$ as the optimal exercise price as it varies with time. This point marks the change from being more profitable to hold to being more profitable to exercise early. Since we do not know what this value is or at what time the best can be found, we are faced with a significantly more difficult problem than in the European option.

We can now split the Black-Scholes equation into two distinct regions and a point. The first region, $0 \leq S < S_f(t)$ gives us

$$P = E - S, \quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP < 0.$$

This is the region where early exercise is optimal. The second, where holding is the best strategy, is found when $S_f(t) < S_\infty$. This corresponds to the PDE

$$P > E - S, \quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0.$$

Finally, we have the condition at $S = S_f(t)$. This gives us the constraints

$$P(S_f(t), t) = \max(E - S_f(t), 0), \quad \frac{\partial P}{\partial S}(S_f(t), t) = -1.$$

We can show that $\frac{\partial P}{\partial S}(S_f(t), t) = -1$ explicitly otherwise we would once again face an arbitrage situation.

At this point, the work relies heavily on [2] and [3]. When we originally approached the problem, several attempts at solving and coding the problem were made. Some were original, and others followed different methods outlined in [4]. However, after the sources [2] and [3], we arrived at a deeper understanding of the problem.

Our original attempts failed to capture the importance of the free-boundary problem and the methods outlined in [4] did not deal with them properly. Once these two alternate sources were found though, they provided an explanation of the problem, as well as methods to solve it. At this stage we used the two sources as a guidebook, and followed their suggestions.

Unfortunately, the books were designed more for financially inclined people and sometimes skim past important mathematical details, such as the convergence of the Projected Successive Over-Relaxation (SOR) method described in Section 4.1. We were unable to find an explanation in the sources we had or alternate sources that could explain the methods in greater detail.

There are several methods that can now be used to solve the American option. Binomial and trinomial trees can be used to approximate a random walk which follows the general tendencies of the asset [3]. That is, the overall value of the stock will grow at a rate consistent with expectations but at any given time step there will be a probability for it to decrease or increase in value by a given amount.

However, one of the most common methods that is presently used is using finite differences to make a mesh for the PDE, and then solving for the values using standard finite difference approximations. As we have seen, there are many different relations that can be used, such as forward difference (explicit), backward difference (implicit), Crank-Nicolson (implicit), and others. Any one of these methods could be used to find a solution, but they have their individual advantages and disadvantages.

The forward difference method is a very simple technique that is easy to implement, but it has an inherent instability that can easily cause solution values to diverge from the proper number. The backward difference method does much to correct the instability issue. The Crank-Nicolson (CN) method is an average of the backward and forward difference methods. It is a very stable method that is second-order accurate. Luckily, [2] provides a method to implement the CN method with the transformation of variables found in Section 2.2. We can see from [2] that the CN method actually becomes quite simple to implement and offers better accuracy and stability over the other two methods. Using MATLAB specific techniques, such as sparse matrices and the backslash operator, we can keep the runtime to a very acceptable level.

When we choose the CN approach, we also find that it is much simpler to transform the PDE into another form [3]. This step of the process is to find a linear complementarity problem for the American put. By doing so we remove the free-boundary, which makes a solution much easier to compute. After we solve the analogous problem, we can recover the free-boundary and the relevant solutions. Using the transformation of variables discussed in (7), we can do just this.

The transformation for the optimal exercise boundary becomes $x = x_f(\tau)$. Since we assumed $S_f(t) < 0$ we also have $x_f(\tau) < 0$. The payoff function becomes

$$g(x,\tau) = e^{\frac{1}{2}(k+1)^2\tau} \max\left(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\right). \tag{10}$$

This gives us the PDE

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } x > x_f(\tau) \tag{11}$$

$$u(x,\tau) = g(x,\tau) \quad \text{for } x \le x_f(\tau) \tag{12}$$

and initial condition

$$u(x,0) = g(x,0) = \max\left(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\right) \tag{13}$$

and the limit

$$\lim_{x\to\infty} u(x,\tau) = 0. \tag{14}$$

We also assume that $u$ and $\frac{\partial u}{\partial x}$ are continuous.

Transforming this into linear complementarity form gives us

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \cdot (u(x,\tau) - g(x,\tau)) = 0, \tag{15}$$

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \ge 0, \quad (u(x,\tau) - g(x,\tau)) \ge 0. \tag{16}$$

This set of equations is also accompanied by the initial and boundary conditions

$$u(x,0) = g(x,0) \tag{17}$$

$$u(-x^-,\tau) = g(-x^-,\tau) = u(x^+,\tau) = g(x^+,\tau) = 0 \tag{18}$$

where $-x^-$ and $x^+$ are the endpoints of the interval under consideration. Now we are at a point where we can begin to solve the problem.

# 4 MATLAB Implementation

As we know, we can easily divide up a space-time problem into a mesh grid. This allows us to discretize the problem and use common scientific computing methods to find a solution. For the solution algorithm, we use a combination of the Crank-Nicolson and SOR algorithms. Let $u_n^m$ denote the solution at $u(n\delta x, m\delta\tau)$. Let $x = n\delta x$ for $N^- \leq n \leq N^+$ and $\tau = m\delta\tau$ . Using the Crank-Nicolson method we have

$$\frac{\partial u}{\partial \tau}(x, \tau + \delta\tau/2) = \frac{u_n^{m+1} - u_n^m}{\delta\tau} + O(\delta\tau^2)$$

and

$$\frac{\partial^2 u}{\partial x^2}(x, \tau + \delta\tau/2) = \frac{1}{2}\left(\frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{\delta x^2}\right) \tag{19}$$

$$+ \frac{1}{2}\left(\frac{u_{n+1}^{m} - 2u_n^{m} + u_{n-1}^{m}}{\delta x^2}\right) + O(\delta x^2). \tag{20}$$

Dropping our error terms, the first inequality (15) becomes

$$u_n^{m+1} - \frac{1}{2}\alpha(u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}) \geq u_n^m + \frac{1}{2}\alpha(u_{n+1}^{m} - 2u_n^{m} + u_{n-1}^{m}) \tag{21}$$

where $\alpha = \frac{\delta\tau}{\delta x^2}$. This also gives us the general form of the payoff function, $g_n^m = g(n\delta x, m\delta\tau)$. For the boundary conditions we have

$$u_{N^-}^m = g_{N^-}^m, \quad u_{N^+}^m = g_{N^+}^m, \quad u_n^0 = g_n^0. \tag{22}$$

If we let $Z_n^m$ be

$$Z_n^m = (1-\alpha)u_n^m + \frac{1}{2}\alpha(u_{n+1}^m + u_{n-1}^m) \tag{23}$$

then 21 becomes

$$(1+\alpha)u_n^{m+1} - \frac{1}{2}\alpha(u_{n+1}^{m+1} + u_{n-1}^{m+1}) \geq Z_n^m. \tag{24}$$

We can now turn this formulation into a linear algebra problem. Let

$$\mathbf{u}^m = \begin{pmatrix} u_{N^-+1}^m \\ \vdots \\ u_{N^+-1}^m \end{pmatrix}, \quad \mathbf{g}^m = \begin{pmatrix} g_{N^-+1}^m \\ \vdots \\ g_{N^+-1}^m \end{pmatrix}. \tag{25}$$

Note that we do not need to include the terms $u_{N^-}^m$ and $u_{N^+}^m$ as they are explicitly defined by the boundary conditions in (22). Let $\mathbf{b}^m$ be defined by

$$\mathbf{b}^m = \begin{pmatrix} b_{N^-+1}^m \\ \vdots \\ b_0^m \\ \vdots \\ b_{N^+-1}^m \end{pmatrix} = \begin{pmatrix} Z_{N^-+1}^m \\ \vdots \\ Z_0^m \\ \vdots \\ Z_{N^+-1}^m \end{pmatrix} + \frac{1}{2}\alpha \begin{pmatrix} g_{N^-}^{m+1} \\ 0 \\ \vdots \\ 0 \\ g_{N^+}^{m+1} \end{pmatrix} \tag{26}$$

We can now begin to see how the matrix operations of the problem will take shape. The Crank-Nicolson method gives us a $(N^+ - N^- - 1)$-square, tridiagonal, symmetric matrix $\mathbf{C}$ which is found to be

$$C = \begin{pmatrix} 1+\alpha & -\frac{1}{2}\alpha & 0 & \dots & 0 \\ -\frac{1}{2}\alpha & 1+\alpha & -\frac{1}{2}\alpha & & \vdots \\ 0 & -\frac{1}{2}\alpha & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1+\alpha & -\frac{1}{2}\alpha \\ 0 & \dots & 0 & -\frac{1}{2}\alpha & 1+\alpha \end{pmatrix} \tag{27}$$

This lets us rewrite our linear complementarity problem (15) into

$$\mathbf{C}\mathbf{u}^{m+1} \geq \mathbf{b}^m, \quad \mathbf{u}^{m+1} \geq \mathbf{g}^{m+1}, \quad (\mathbf{u}^{m+1} - \mathbf{g}^{m+1}) \cdot (\mathbf{C}\mathbf{u}^{m+1} - \mathbf{b}^m) = 0. \tag{28}$$

Now we see where the time-steps arise: each vector $\mathbf{u}^{m+1}$ can be calculated from $\mathbf{b}^m$. With each $\mathbf{u}^m$ we can calculate $\mathbf{b}^m$. To step through these loops, we use the projected SOR method.

## 4.1 The Projected SOR Method

There are two main approaches to the time-stepping portion of the grid, which are outlined in [2]. The first is to solve the linear system by using a technique such as the LU-decomposition. This method allows for very quick methods of solving the system for the European option. However, this method, which is a direct method, is difficult to transform to the American option and nonlinear problems which could include transaction costs [3].

Our other option is an iterative method, of which there are several. These methods use a guess which is gradually improved to reach the exact solution. While these are slower than a direct method for the same problem, such as a European option, they are easily extendable to the American option [2]. Two iterative methods are the Jacobi and Gauss-Seidel methods. Both are used to iterate through to the exact solution. However, the Projected SOR method, which is actually a refinement of the Gauss-Seidel method [3], can significantly speed up the rate of convergence and require many less iterations. This significantly reduces runtime [3].

Herein lies one of the foggy parts of the methods outlined in [2]. As Professor Yong has mentioned, the LU method, as well as the iterative methods mentioned above, are used to solve linear systems of equations. By transforming the Black-Scholes equation to the linear complementarity problem, [2] appears to use the SOR method to solve it. They state that these two problems are equivalent, but unfortunately offer no proof, as it is beyond the scope of their book. In the end, we decided to use the SOR method on the word of [2] and [3] as it provided what appeared to be the best approach to solve the problem.

As mentioned, we use an iterative time-stepping algorithm to find a proper solution for $u_n^m$ from [2]. In this case let $u_n^{m,k}$ be defined as the $k$th iterate of $u_n^m$. This implies that the next solution would be $u_n^{m,k+1}$. Let the initial guess for $u_n^{m+1}$ be the value from the previous step, namely $u_n^m$. This gives us that $u_n^{m+1,0} = u_n^m$. As $k \to \infty$, $u_n^{m,k} \to u_n^m$.

Now we can see that, as defined above, $u_n^{m,k} \to u_n^m$ as $k$ increases. However, by multiplying by a scalar greater than one, we can significantly speed up the convergence. This is the idea behind the SOR method. From the Crank-Nicolson method we can let

$$y_n^{m+1,k+1} = \frac{1}{1+\alpha}\left(b_n^m + \frac{1}{2}\alpha(u_{n-1}^{m+1,k+1} + u_{n+1}^{m+1,k})\right). \tag{29}$$

Then we have

$$u_n^{m+1,k+1} = \max(u_n^{m+1,k} + \omega(y_n^{m+1,k+1} - u_n^{m+1,k}), g_n^{m+1}). \tag{30}$$

In this case, $\omega(y_n^{m+1,k+1} - u_n^{m+1,k})$ can be thought of as a correction factor, that helps to converge onto $u_n^{m,k+1}$. By letting $1 < \omega < 2$ we over-correct and speed up the process. Once the factor being added each step becomes significantly small, we say the solution has converged and return the value for $u_n^{m,k+1}$.

This is also the point where we find the differences between American, Bermudan, and European options. In equation 30, we are determining whether it is better to exercise the option early or to hold it by taking the max of the two values. For the American option, we always make this comparison whereas for the European we never do. The Bermudan is once again a mixture, where we make the comparison on specified time steps.

The code implementing both the Crank-Nicolson and Projected SOR methods as outlined in [2] can be found in the appendix.

## 4.2 Results

By using built in techniques that look to optimize performance, such as sparse matrices, the code is incredibly efficient. When we let $N = 1000$ and $M = 800$, a solution is returned in less than half a second. They are also very accurate. Let us consider an American put with a strike price of \$10, $r = 0.1$, $\sigma = 0.4$, with times to maturity of three and six months.

As we can see in the first chart 4.2, overall performance compared to the book [2] is quite good. The largest error, found when the asset price is \$16 with

| Asset Price | Payoff Value | $\frac{\text{put.m}}{(3 \text{ months})}$ | Book | $\frac{\text{put.m}}{(6 \text{ months})}$ | Book |
|---|---|---|---|---|---|
| 2.00 | 8.0000 | 8.0023 | 8.0000 | 8.0023 | 8.0000 |
| 4.00 | 6.0000 | 5.9971 | 6.0000 | 5.9971 | 6.0000 |
| 6.00 | 4.0000 | 4.0019 | 4.0000 | 4.0019 | 4.0000 |
| 8.00 | 2.0000 | 2.0230 | 2.0200 | 2.0970 | 2.0951 |
| 10.00 | 0.0000 | 0.6871 | 0.6913 | 0.9156 | 0.9211 |
| 12.00 | 0.0000 | 0.1734 | 0.1711 | 0.3642 | 0.3622 |
| 14.00 | 0.0000 | 0.0330 | 0.0332 | 0.1312 | 0.1320 |
| 16.00 | 0.0000 | 0.0055 | 0.0053 | 0.0448 | 0.0460 |

Figure 1: Chart 1

| Asset Price | Payoff Value | $\frac{\text{put.m}}{(3 \text{ months})}$ | Exact | $\frac{\text{put.m}}{(6 \text{ months})}$ | Exact |
|---|---|---|---|---|---|
| 2.00 | 8.0000 | 7.7554 | 7.7531 | 7.5161 | 7.5123 |
| 4.00 | 6.0000 | 5.7503 | 5.7531 | 5.5104 | 5.5128 |
| 6.00 | 4.0000 | 3.7589 | 3.7569 | 3.5604 | 3.5583 |
| 8.00 | 2.0000 | 1.9049 | 1.9024 | 1.9198 | 1.9181 |
| 10.00 | 0.0000 | 0.6644 | 0.6694 | 0.8651 | 0.8703 |
| 12.00 | 0.0000 | 0.1696 | 0.1675 | 0.3499 | 0.3477 |
| 14.00 | 0.0000 | 0.0325 | 0.0326 | 0.1273 | 0.1279 |
| 16.00 | 0.0000 | 0.0053 | 0.0054 | 0.0437 | 0.0448 |

Figure 2: Chart 2

three months to maturity, is 3.6%. The average error for the entire chart is 0.7%.

We can also compare out results for the European option with those calculated directly from the closed form solution in the second chart 4.2. For the same specifications as above but simply a European option we find that our results are once again very accurate. The average error for the entire chart is 0.6% with the largest error found at an asset price of $16 with three months to go of 2.5%.

### 4.2.1 Order of Accuracy

We should note that this project led to an interesting finding. As mentioned above, we should have a second-order of accuracy for the spatial step, and a first-order for the time step. But when we set one step size to be constant and decrease the other, the numerics do not quite work out this way.

To determine the order of accuracy, we chose a standard value for the parameter not being varied. This value ended up being 320, as it allowed plenty of steps for the time-stepping portion to converge and also a fine enough grid for the spatial step to determine the exact price to the necessary degree. Beginning with 10 steps, each iteration doubled the number of steps until the final number of 1280 was reached. To calculate the error in the system, each iteration compared the calculated value for nine stock prices (one equal to the exercise price and then four on each side, above and below, at \$2 increments) to the calculated values provided by the closed form solution of the European option. The error over these nine values was averaged, and then entered in as a data point.

We can then use log plots and polynomial fits to determine the slope of the line, and thus the order of accuracy of the system. For this particular experiment, $N$ and $M$ varied as described above, strike price was set to \$10, $r = 0.1$, $\sigma = 0.4$, and time to expiration was three months. We find that the spatial is a first-order, with an exact somewhere between 1 and 1.5, and that the temporal is significantly less, along .25 to .75. This is a rather strange discovery. At present, we have yet to determine any reason why these convergence rates would differ so much from their supposed values. A deeper investigation into the algorithm could possibly produce a solution.

By achieving an accurate result with a very short runtime, we have accomplished our goal for this project.

# 5 Future Work

There are a number of ways that this project could be continued. Besides the American put, there are many different sorts of options that could be explored. These are often called "Exotic" options, and they introduce myriad new avenues for research. Also, an exploration into even deeper, more precise approximations could yield an even more exact solution.

We should also look into the issue of order of accuracy. At present, we have yet to determine why the calculated values are different than the theoretical ones.

These are all reasonable procedures that could be undertaken in the future.

## References

[1] Black, F. & Scholes, M. (1973) *The pricing of options and corporate liabilities*. J. Pol. Econ. **81** 637-659.

[2] Wilmott, P. & Howison, S. & Dewynne, J. (1995) *The Mathematics of Financial Derivatives*. Cambridge University Press.

[3] Wilmott, P. & Howison, S. & Dewynne, J. (1993) *Option Pricing*. Oxford Financial Press.

[4] Sukha, S. (2001) *Finite-Difference Methods for Pricing the American Put Option*. Advanced Mathematics of Finance Honours Project.