Pricing options

American, Bermudan, and European Put Options

Three Types of Options

- European
  - Most basic
  - Can only be exercised on the exercise date
  - Closed form solution long since discovered
- Bermudan
  - In between European and American
  - Can be exercised on set dates, usually once a month
- American
  - Most complicated
  - Can be exercised at any time
    - Free boundary problem

THE BLACK-SCHOLES EQUATION

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

V = Option value
S = Stock price

σ = Stock volatility
r = Risk-free return rate

We'll be focusing on Put options, so let V = P from now on.

Boundary and Initial Conditions

$$P(S,T) = \max(E - S, 0)$$

$$P(0,t) = Ee^{-r(T-t)}$$

$$P(S,t) \to 0 \text{ as } S \to \infty$$

T = Exercise date
E = Strike price

The American Option

- The free boundary leads to inequalities, rather than strict equalities

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \leq 0$$

- Two regions
  - Optimal to exercise: $0 \leq S \leq S_f(t)$, $P = E - S$
  - Optimal to hold: $S_f(t) \leq S \leq \infty$, $P > E - S$
  - At the change: $P\left(S_f(t), t\right) = \max\left(E - S_f(t), 0\right)$, $\frac{\partial P}{\partial S}\left(S_f(t), t\right) = -1$

LINEAR COMPLIMENTARITY

- Change of Variables

$$S = Ee^{x}, \qquad t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, \qquad P = Ev(x,\tau)$$

- Through algebra, we get the equivalent problem

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(u(x,\tau) - g(x,\tau)) = 0$$

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \geq 0, \qquad (u(x,\tau) - g(x,\tau)) \geq 0$$

$$u(x,0) = g(x,0), u(-x^-,\tau) = g(-x^-,\tau), u(x^+,\tau) = g(x^+,\tau)$$

$$g(x,\tau) = e^{\frac{1}{2}(k+1)^2\tau} \max\left(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\right)$$

CRANK-NICOLSON METHOD

- Sweet order of accuracy
- Gives us the system

$$u_n^{m+1} + \frac{1}{2}\alpha(u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}) \geq u_n^m + \frac{1}{2}\alpha(u_{n+1}^m - 2u_n^m + u_{n-1}^m)$$

- First we solve

$$Z_n^m = u_n^m + \frac{1}{2}\alpha(u_{n+1}^m - 2u_n^m + u_{n-1}^m)$$

- And then…

The Projected SOR Iterative Method

- SOR = Successive Over Relaxation
- Using the relation

$$u_n^{m+1} + \frac{1}{2}\alpha(u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}) \geq Z_n^m$$

- We get

$$y_n^{m+1,k+1} = \frac{1}{1+\alpha}\left(Z_n^m + \frac{1}{2}\alpha(u_{n+1}^{m+1,k+1} + u_{n-1}^{m+1,k+1})\right)$$

  - Where $y_n^{m+1,k+1}$ is the *k*+*1*th iterate
- This converges to the desired solution using standard tolerance bounds

THE DIFFERENCE

- American

$$u_n^{m+1,k+1} = \max\left(u_n^{m+1,k} + \omega\left(y_n^{m+1,k+1} - u_n^{m+1,k}\right), g_n^{m+1}\right)$$

- European

$$u_n^{m+1,k+1} = u_n^{m+1,k} + \omega\left(y_n^{m+1,k+1} - u_n^{m+1,k}\right)$$

- Bermudan
  - Takes on both forms depending if the time step allows for early exercise

Results

- E = 10, r = .1, σ = .4, T = 3 months
  - American

Stock Value	Option Price	Book Answer	Error
2.00	8.0023	8.0000	0.0287
4.00	5.9971	6.0000	0.0483
6.00	4.0019	4.0000	0.0475
8.00	2.0230	2.0200	0.1485
10.00	0.6871	0.6913	0.6076
12.00	0.1734	0.1711	1.3442
14.00	0.0330	0.0332	0.6024
16.00	0.0053	0.0055	3.6364

Results

- $E = 10$, $r = .1$, $\sigma = .4$, $T = 3$ months
  - European

Stock Value	Option Price	Exact Price	Error
2.00	7.7554	7.7531	0.0302
4.00	5.7503	5.7531	0.0486
6.00	3.7589	3.7569	0.053
8.00	1.9049	1.9024	0.1281
10.00	0.6644	0.6694	0.7454
12.00	0.1696	0.1675	1.2382
14.00	0.0325	0.0326	0.3224
16.00	0.0053	0.0054	2.2386

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