Modeling a Particle in an Electrostatic Analyzer

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1 Background

In 2011, NASA is planning to launch the Juno spacecraft to Jupiter. Among many instruments on board the craft, there will be three Electrostatic Analyzers (ESAs), instruments designed to measure the flux of electrons in the space around the craft. The ESAs will be used to gain better understanding into the auroras of Jupiter, which would shed some light on Jupiter's internal dynamics. A cross-section of the Juno ESA is shown in Figure 1. This ESA has two sets

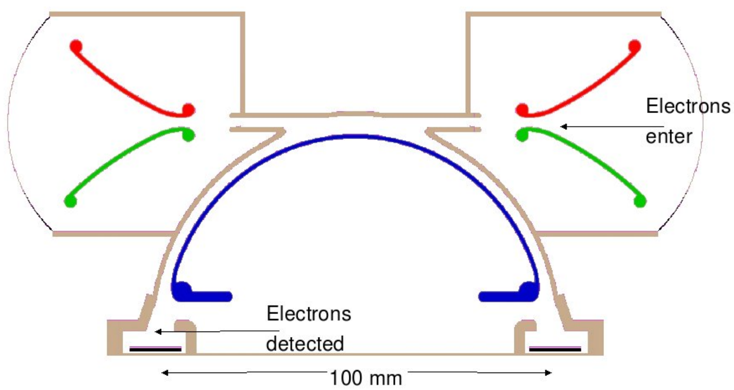

Figure 1: Electrostatic analyzer. Voltages used to sort electrons are applied to the deflection plates, shown in green and red, and to the hemispherical dome, shown in blue.

of adjustable voltages, one to select out the energies of the electrons and one to select the incoming angle. The ESA also has a fairly complicated geometry, including shielding, structural elements, and detection plates.

The ESA geometry is complicated enough to make it impossible to solve analytically. One of the 2007-2008 clinic teams, the Math/Physics clinic sponsored by Southwest Research Institute, is working on analyzing the ESA numerically, in the presence of magnetic fields; the starting point for that work is the output from the SIMION$^{\text{TM}}$simulation suite. SIMION$^{\text{TM}}$is a software package for simulating the motion of charged particles; a screenshot of the SIMION^{TM}interface is shown in Figure 2.

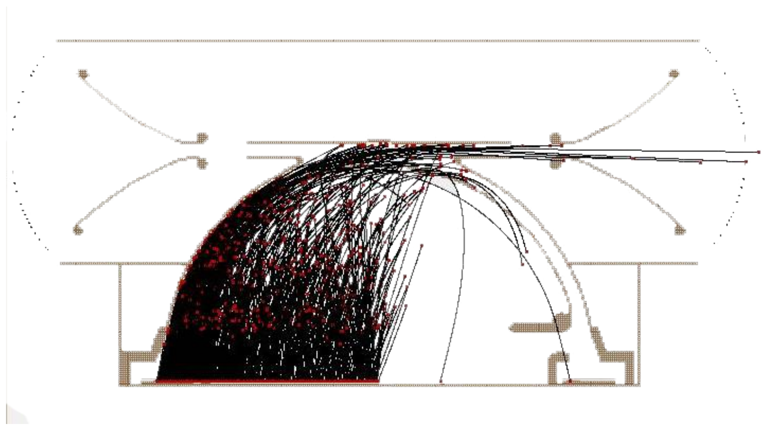

Figure 2: Sample trajectories in an ESA, as calculated by SIMION^{TM}

2 Problem Statement

For this project, I will attempt to calculate the trajectories of the particles in a simplified version of an ESA, as is done by SIMION^{TM}. I will take into account fringing fields and some of the nonidealities of the instrument, though I will not attempt to duplicate its entire geometry. This could be used to create a model for the energies of detected electrons in a non-ideal ESA. Creating this model in the presence of magnetic fields is the clinic project for the SwRI Math/Physics clinic team; for the Scientific Computing final project, the goal is merely to be able to calculate trajectories.

3 Analytical Treatment

The simplest model for the ESA is shown in Figure 3. The electrons are launched on the left side, and are assumed to be 'detected' if they can go through the semicircle and get to the other side without impacting any surfaces. The core of the ESA consists of a hemispherical plate of radius r_1 with a voltage V on it. We can assume that the gap $r_2 - r_1$ between the hemispherical plate and the grounded outer wall is small compared to the average radius $\bar{r}$, $r_2 - r_1 << \bar{r}$. Thus, in the gap, the field is approximately constant in magnitude and radial. It is given by $\vec{E} = \frac{V}{r_2 - r_1}\hat{r}$. A charged particle inside the ESA will experience a uniform radial force of $\frac{qV}{r_2 - r_1}\hat{r}$. If this force is just right, the particle will be accelerated centripetally, and will be able to get through the ESA and to the detector; if the particle is moving too fast or too slow, it will crash into either

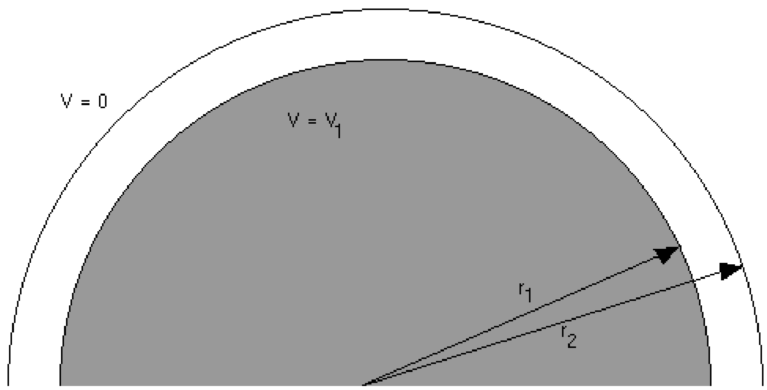

Figure 3: The simplest possible ESA - two concentric circles, with the inner one having a nonzero potential V_1.

the inner or outer wall and will be lost. We can see that

$$
\begin{aligned}
F &= \frac{qV}{r_2 - r_1} \\
F &= -\frac{mv^2}{\bar{r}} \\
\frac{qV}{r_2 - r_1} &= -\frac{mv^2}{\bar{r}} \\
\frac{1}{2}mv^2 &= \frac{-qV\bar{r}}{2(r_2 - r_1} \\
v &= \sqrt{\frac{-qV\bar{r}}{m(r_2 - r_1)}}
\end{aligned}
$$

Thus, we can calculate that given a voltage V on the inner plate, particles of energy $T = \frac{-\bar{r}qV}{2(r_2 - r_1)}$ and, thus, initial velocity $\sqrt{\frac{-\bar{r}qV}{m(r_2 - r_1)}}$ will make it through and be detected.

4 Nonidealities and modifications

This situation is fairly well-understood; however, a real ESA has a number of geometric nonidealities, which would break these simple relationships. The nonideality I am interested in introducing is the path for the electron to enter the ESA. In a "top-hat" ESA, the entrance point for the electrons is at the top, and electrons can enter from any direction.

The ESA I am simulating is shown in Figure 4. As before, there is an inner curved plate of radius r_1, and an outer curved plate of radius r_2. However, the outer plate now has a thickness, and is solid between r_2 and r_3. At the center,

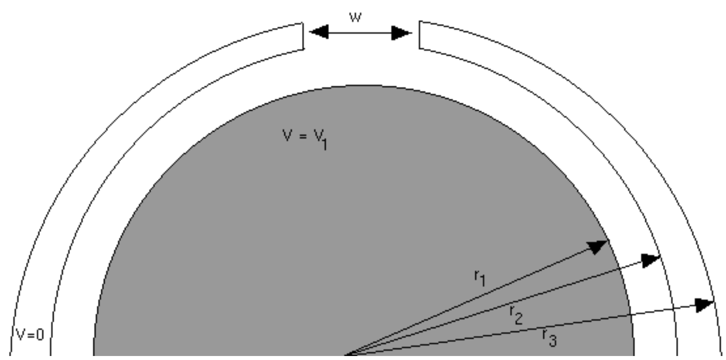

Figure 4: An ESA which now has a gap at the top for electrons to enter. The outer plate also has a nonzero thickness.

it has a gap of width w to let the electrons escape. An electron is considered to be 'detected' if it can traverse a path from the bottom of the ESA and out the gap. Because of the time-reversibility of the underlying physical equations, this is equivalent to an electron being able to travel from the outside in. However, the reverse situation, paths from the inside out, are significantly easier to work with.

5 Algorithms

The first portion of the simulation requires calculating the electric fields throughout the device. This is done by using the method of relaxation. The method of relaxation is a way of numerically solving Laplace's equation, $\nabla^2 V = 0$. It relies on the fact that solutions to Laplace's equation have the property that the value of the function at a point is equal to the average value of the function over the points at distance r away. The method of relaxation involves iteratively setting the value of each point equal to the average of its neighbors; this will eventually converge to an approximate solution.

For the given problem, electric fields are calculated using the method of relaxation on the electric potential. The space around the ESA is discretized by creating points equally spaced throughout the ESA and the volume around it. All points within the inner radius of the ESA, $r < r_1$, are assigned to a poential of V_i. All points within the outer wall, specified by $r_2 < r < r_2$ and $|x| > \frac{w}{2}$, are set to zero. The points on the outer boundary of the region of interest are also set to zero, implementing the boundary condition that the field must go to zero at infinity. These points are marked as fixed, so that they are not changed by the subsequent relaxation. The points on the interior of the ESA, $r_1 < r < r_2$, are initialized with a guess at the potential, using the simple linear approximation, $V = V_i(1 - \frac{r - r_1}{r_2 - r_1})$. Though this is not necessary for the algorithm to converge, it significantly speeds up the convergence rate. These points are not fixed.

When a single step of relaxation is done, a new matrix of V-values is created, with the potential at each point being set equal to the average value of its four neighbors, using $V_{new}(i,j) = \frac{1}{4}(V(i,j-1)+V(i,j+1)+V(i-1,j)+V(i+1,j))$ and excluding the points which are kept 'fixed' from the pudate. For points which are designated fixed, the value is kept unchanged, $V_{new}(i,j) = V(i,j)$.

This process is repeated until the values converge. I defined convergence by measuring the maximum change in the potential of any point; once this dropped to below a certain threshhold, meaning that subsequent iterations no longer had an effect, the potential was saved and the relaxation process terminated.

This convergence process can be quite slow. However, it can be accelerated by choosing an appropriate initial guess for the potential. Using a linear approximation as an initial guess speeds up the convergence by two orders of magnitude, since the approximation is correct for most of the space except for the area around the ESA gap.

There are also some inaccuracies arising from the creation of a grid. Since the obstacles are circular and the grid is cartesian, this method essentially creates a jagged edge of the obstacle. The innacuracies are smoothed out within several points of the edge, but at the very edge, the potential is inaccurate.

Next, the electric field needs to be calculated from the electric potential. This involves taking the gradient; the function is specified on discrete points, so using any one of the derivative formulas could work. I chose to use a second-order formula on the interior, setting $\frac{dV}{dx}(i,j) = \frac{V(i+1,j)-V_(i-1)}{2\delta x}$. The derivative in the y direction was calculated analogously. On the edges, I used a first-order accurate forward-difference method, but this choice is not significant; the electric field along the edges in the perpendicular direction is zero anyway, and the method by which this zero or almost-zero number number is calculated is not significant.

This then gives matrices containing the electric field componetns in the x and y directions. However, this is not sufficient; constraining particles to move along a grid would be extremely restrictive, and thus the electric field at off-grid points needs to be calculated. This is done by merely interpolating the electric field linearly between adjacent gridpoints.

Lastly, the trajectory of a particle needs to be calculated. This was done using the leapfrog method, a second-order accurate algorithm, which sets

$$\begin{aligned} x_{n+1} &= x_n + v_n\delta t + \frac{1}{2}a_n(\delta t)^2 \\ v_{n+1} &= v_n + \frac{a_n + a_{n+1}}{2} \end{aligned}$$

with the accelerations calculated using

$$\vec{a} = \frac{\vec{F}}{m} = \frac{q\vec{E}}{m}$$

6 Results

The first step is to verify the calculation of the electric fields. With a voltage of $100V$ on the inner plates, an inner ESA radius of $r_1 = 45mm$, an outer ESA radius of $r_2 = 50mm$, an outer thickness of 1 giving $r_3 = 51mm$, and a gap width of 15 mm, fringe fields are clearly visible. A contour plot of the potential is shown in Figure 5. We can see that around the gap, the potential extends outward, and takes more space to drop to zero, and so we would expect the electric fields to be lower in that area.

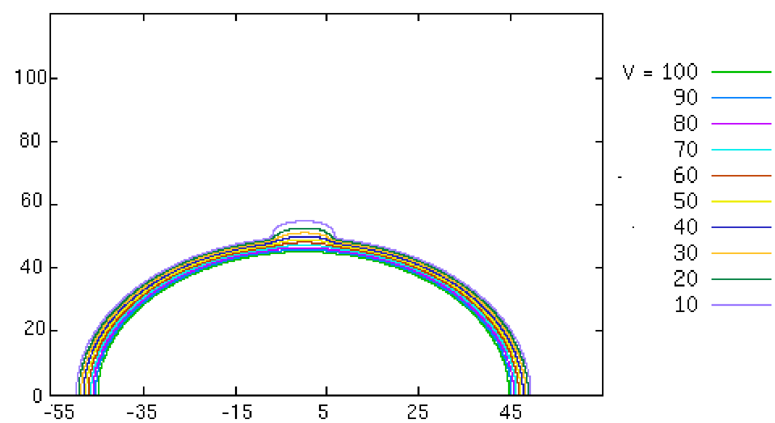

Figure 5: Contour plot of the electric potential of the nonideal ESA. Fringing is clearly visible. Potential is in V, distances are in mm.

Given the values for the ESA described above and setting a particle charge and mass equal to the charge and mass of an electron, we would expect that a particle with initial velocity of $1.36 \times 10^7 m/s$ would be able to travel around the ESA. Shooting a particle through at that energy does not work. A plot of the radial position of the particle versus the x-position of the particle is shown in Figure 6. The particle starts out at the average radius, 47.5, and at the far left end of the ESA, at $x = -47.5,\ y = 0$. However, the particle is moving a little bit too fast - the radial position of the particle increases as the particle moves trhough the ESA. As the particle gets closer to $x = 0$, closer to the center of the nonideality, the radius increases faster - because of the gap, the field is weaker than expected.

However, it is possible to have a particle with slightly different velocity escape from the ESA through the gap. An example trajectory of an escaping particle is shown in Figure 7. The electron comes out at an angle; the minimum angle is determined by the thickness of the outer shell of the ESA and the width of the gap. the radial position of the particle is shown in Figure 8. The radial position of an escaping particle first decreases and then increases; to escape, the particle approaches the inner surface of the ESA but then escapes when the electric field weakens when it passes underneath the gap. The initial velocity of this particle

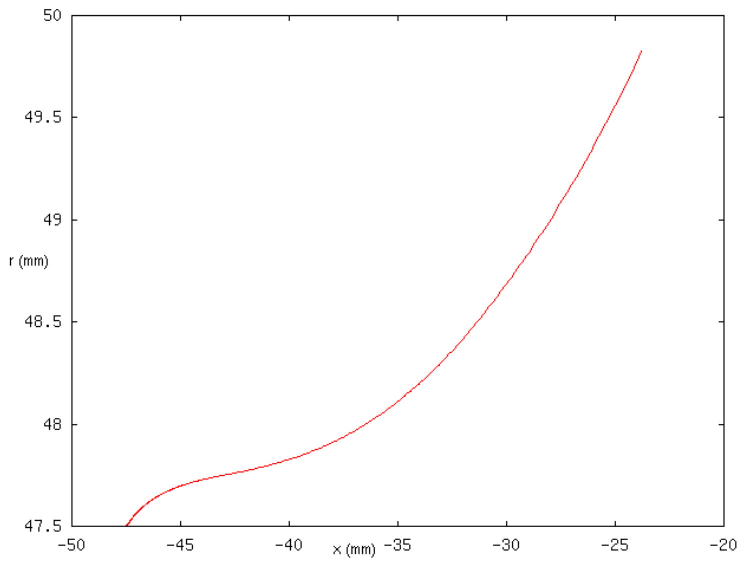

Figure 6: Radial position of the electron versus the x position of the particle. The outer radius of the ESA is 50 mm and the inner radius is 45 mm; the entrance gap has width 15 mm.

is $1.232 \times 10^7 m/s$, a little bit lower than the expected escape velocity for an ideal ESA. This makes sense, because we can see that to escape, the particle needs to be in a state where, in the area where fringe fields are not affecting it, it would be slowly getting closer to the center of the ESA. On Figures 7 and 8, you can see the three states it passes through - at first, it is moving approximately circularly, which can be seen in the xy plot; however, looking at the xr plot, we can see that its radial position is decreasing. Then, on the xr plot, we can see the change to an increasing radius as it passes through the gap; lastly, on both plots, there is visible a large tail at the end, when the particle is moving away from the ESA on the outside.

There is a small range of velocities with which a particle can escape, given the initial position and direction; even without varying the initial position and angle, assigning the particle to move vertically at the start of its trajectory and starting at $x = -47$, particles with velocities off by up to half a percent can escape. The trajectories look the same, so I have not included additional plots. If the initial position and angle is allowed to vary, the range of allowed velocities would also increase.

Trying to make a particle with those initial constraints go around the entire ESA turns out to be impossible; a significantly different initial condition is necessary to allow the particle to navigate the nonideality of the ESA. A script to optimize the distance traveled by the electron using the bisection method yielded 1.1644×10^7 as the optimal energy; a plot of the radial position versus

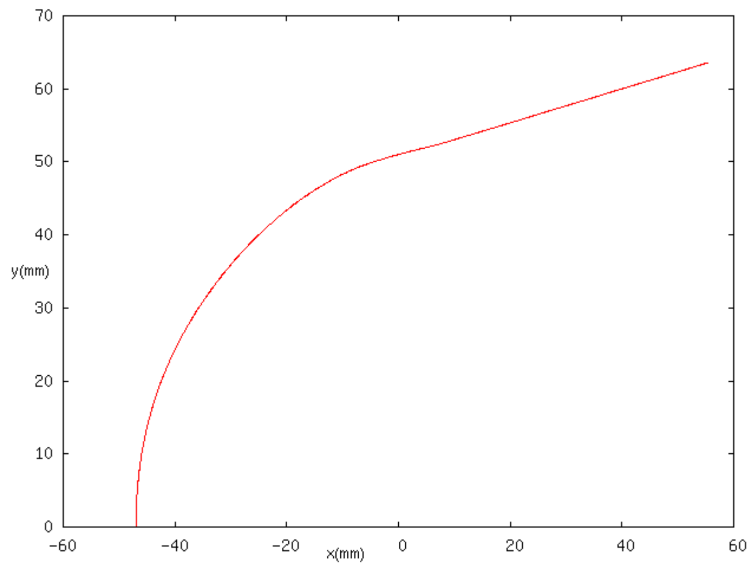

Figure 7: Trajectory of an electron which manages to escape the ESA. It moves in a circle until it escapes, around $x = 0$, at which point it moves approximately linearly.

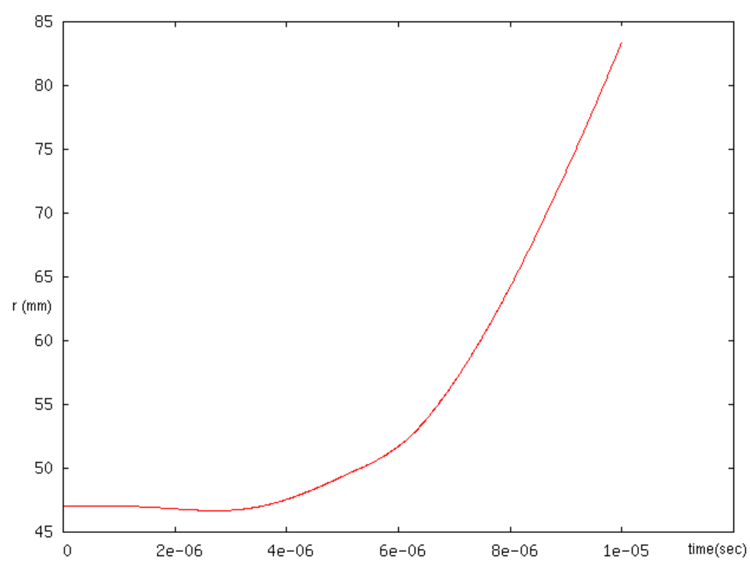

Figure 8: Trajectory of an electron which manages to escape the ESA.

the x-coordinate of the particle is shown in Figure 9. The gap causes enough of a decrease in the electric field so that even a particle with a low enough energy to barely avoid hitting the inner plate by the time it gets to the gap will hit the outer plate soon after passing past the gap.

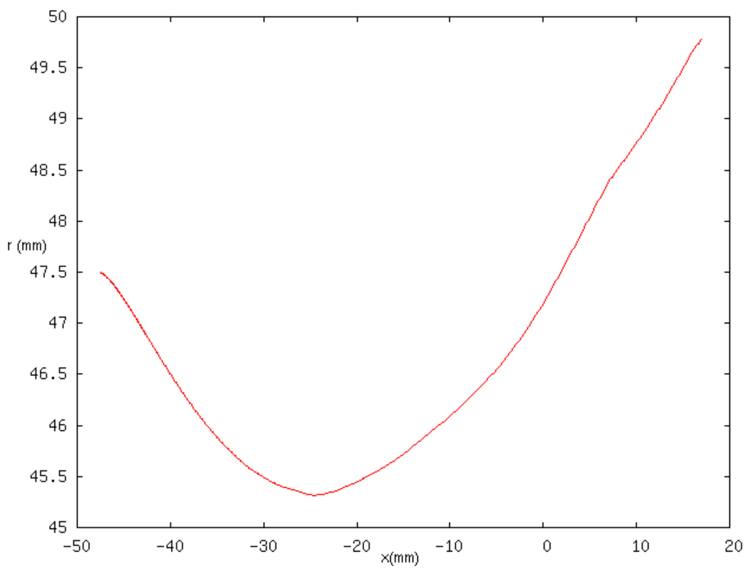

Figure 9: This is what seems to be the farthest an eletron can get, if it starts at the middle of the ESA gap moving vertically.