

# Spectral Methods for Numerical Relativity

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## Abstract

Equations arising in General Relativity are usually too complicated to be solved analytically and one has to rely on numerical methods to solve sets of coupled, partial differential, equations. Amongst the possible choices, this paper focuses on a class called spectral methods where, typically, the various functions are expanded onto sets of orthogonal polynomials or functions. A theoretical introduction on spectral expansion is first given and a particular emphasis is put on the fast convergence of the spectral approximation. We present then different approaches to solve partial differential equations, first limiting ourselves to the one-dimensional case, with one or several domains. Generalization to more dimensions is then discussed. In particular, the case of time evolutions is carefully studied and the stability of such evolutions investigated. One then turns to results obtained by various groups in the field of General Relativity by means of spectral methods. First, works which do not involve explicit time-evolutions are discussed, going to rapidly rotating strange stars to the computation of binary black holes initial data. Finally, the evolutions of various systems of astrophysical interest are presented, from supernovae core collapse to binary black hole mergers.

# 1 Introduction

Einstein's equations represent a complicated set of nonlinear partial differential equations for which some exact [23] or approximate [24] analytical solutions are known. But these solutions are not always suitable for some physically or astrophysically interesting systems, that require an accurate description of their relativistic gravitational field, without any assumption on the symmetry or with the presence of matter fields for instance. Therefore, many efforts have been undertaken to solve Einstein's equations with the help of computers in order to model relativistic astrophysical objects. Within this field of *numerical relativity*, several numerical methods have been experimented and a large variety of them are currently being used. Among them, *spectral methods* are now increasingly popular and the goal of this review is to give an overview (at the moment it is written or updated) of the methods themselves, the groups using them and the obtained results. Although some theoretical framework of spectral methods is given in Secs. 2 and 3, more details about spectral methods can be found in the books by Gottlieb and Orszag [68], Canuto *et al.* [44, 45, 46], Fornberg [61] and Boyd [39]. While these references have of course been used for writing this review, they can also help the interested reader to get deeper understanding of the subject. This review is organized as follows: hereafter in the introduction, we briefly introduce the spectral methods, their usage in computational physics and give a simple example. Sec. 2 gives important notions concerning polynomial interpolation and the solution of ordinary differential equations (ODE) with spectral methods. The cases of partial differential equations (PDE), including time evolution or several spatial dimensions, are treated in Sec. 3. The last two sections are then reviewing results obtained using spectral methods: on stationary configurations and initial data (Sec. 4), and on the time-evolution (Sec. 5) of stars, gravitational waves and black holes.

## 1.1 About Spectral Methods

When doing simulations and solving PDE, one faces the problem of representing and manipulating functions on a computer, which deals only with (finite) integers. Let us take a simple example of a function  $f : [-1, 1] \rightarrow \mathbb{R}$ . The most straightforward way to approximate it is through *finite-differences methods*: first one must setup a *grid*

$$\{x_i\}_{i=0\dots N} \subset [-1, 1]$$

of  $N + 1$  points in the interval, and represent  $f$  by its  $N + 1$  values on these grid points

$$\{f_i = f(x_i)\}_{i=0\dots N}.$$

Then, the (approximate) representation of the derivative  $f'$  shall be, for instance

$$\forall i < N, f'_i = f'(x_i) \simeq \frac{f_{i+1} - f_i}{x_{i+1} - x_i}. \quad (1)$$

If we suppose an equidistant grid, so that  $\forall i < N, x_{i+1} - x_i = \Delta x = 1/N$ , the error in the approximation (1) will decay as  $\Delta x$  (first-order scheme). One can imagine higher-order schemes, with more points involved for the computation of each derivative and, for a scheme of order  $n$ , the accuracy will go as  $(\Delta x)^n = 1/N^n$ .

*Spectral methods* represent an alternate way: the function  $f$  is no longer represented through its values on a finite number of grid points, but using its coefficients (coordinates)  $\{c_i\}_{i=0\dots N}$  in a finite basis of known functions  $\{\Phi_i\}_{i=0\dots N}$

$$f(x) \simeq \sum_{i=0}^N c_i \Phi_i(x). \quad (2)$$

A relatively simple case is, for instance, when  $f(x)$  is a periodic function of period 2, and the  $\Phi_i(x) = \cos(\pi i x), \sin(\pi i x)$  are trigonometric functions. Eq. (2) is then nothing but the truncated Fourier decomposition of  $f$ . In general, derivatives can be computed from the  $c_i$ 's, with the knowledge of the expression for each derivative  $\Phi'_i(x)$  as a function of  $\{\Phi_i\}_{i=0\dots N}$ . The decomposition (2) is approximate in the sense that  $\{\Phi_i\}_{i=0\dots N}$  represent a complete basis of some finite-dimension functional space, whereas  $f$  usually belongs to some other infinite-dimension space. Moreover, the coefficients  $c_i$  are computed with finite accuracy. Among the major advantages of using spectral methods is the exponential decay of the error (as  $e^{-N}$ ), for well-behaved functions (see Sec. 2.4.3); one therefore has an *infinite*-order scheme.

In a more formal and mathematical way, it is useful to work with the methods of weighted residuals (MWR, see also Sec. 2.5). As in that section, let us consider the PDE

$$Lu(x) = s(x) \quad x \in U \subset \mathbb{R}^d, \quad (3)$$

$$Bu(y) = 0 \quad x \in \partial U, \quad (4)$$

where  $L$  is a linear operator,  $B$  the operator defining the boundary conditions and  $s$  is a source term. A function  $\bar{u}$  is said to be a *numerical solution* of this PDE if it satisfies the boundary conditions (4) and makes “small” the residual

$$R = L\bar{u} - s. \quad (5)$$

If the solution is searched in a finite-dimensional subspace of some given Hilbert space (any relevant  $L^2_U$  space) in terms of the expansion (2), then the functions  $\{\Phi_i(x)\}_{i=0\dots N}$  are called *trial functions* and, in addition the choice of a set of *test functions*  $\{\chi_i(x)\}_{i=0\dots N}$  defines the notion of smallness for the residual by means of the Hilbert space scalar product

$$\forall i = 0\dots N, \quad (\chi_i, R) = 0. \quad (6)$$

Within this framework, various numerical methods can be classified according to the choice of the trial functions:

- **Finite differences:** the trial functions are overlapping local polynomials of low order,
- **Finite elements:** the trial functions are smooth functions which are non-zero only on subdomains of  $U$ ,
- **Spectral methods:** the trial functions are global smooth functions on  $U$ .

Various choices of the test functions define different types of spectral methods, as detailed in Sec. 2.5. Usual choices for the trial functions are (truncated) Fourier series, spherical harmonics or orthogonal families of polynomials.

## 1.2 Spectral Methods in Physics

We do not give here all the fields of physics where spectral methods are being employed, but we sketch the variety of equations and physical models that have been simulated with such techniques. Spectral methods originally appeared in numerical fluid dynamics, where large spectral hydrodynamics codes have been regularly used to study turbulence and transition, since the seventies. For fully resolved, direct numerical calculations of Navier-Stokes equations, spectral methods were often preferred for their high accuracy. Although high-order finite-difference codes can yield similar accuracy, spectral methods still have an advantage because they permit fast, direct solution of Poisson's equation. Solving Poisson's equation is required to determine the pressure gradient that appears in the Navier-Stokes equations. Historically, they also allowed for two- or three-dimensional

simulations of fluid flows, because of their reasonable computer memory requirements. Many applications of spectral methods in fluid dynamics have been discussed by Canuto *et al.* [44, 46], and the techniques developed in that field can be of some interest for numerical relativity.

From pure fluid-dynamics simulations, spectral methods have rapidly been used in connected fields of research: geophysics [130], meteorology and climate modeling [146]. In this last domain of research, they provide global circulation models that are then used as boundary conditions to more specific (lower-scale) models, with improved micro-physics. In this way, spectral methods are only a part of the global numerical model, combined with other techniques to bring the highest accuracy, for a given computational power. Solution of the Maxwell equations can, of course, be also obtained with spectral methods and therefore, magneto-hydrodynamics (MHD) have been studied with these techniques (see *e.g.* Hollerbach [84]). This has been the case in astrophysics too, where for example spectral three-dimensional numerical models of solar magnetic dynamo action realized by turbulent convection have been computed [42]. Still in astrophysics, the Kompaneet's equation, describing the evolution of photon distribution function in a bath of plasma at thermal equilibrium within the Fokker-Planck approximation, has been solved using spectral methods to model the X-ray emission of *Her X-1* [26, 32]. In the simulations of cosmological structure formation or galaxy evolution, many N-body codes rely on a spectral solver for the computation of the gravitational force by the so-called particle-mesh algorithm. The mass corresponding to each particle is decomposed onto neighboring grid points, thus defining a density field. The Poisson equation giving the Newtonian gravitational potential is then usually solved in Fourier space for both fields [83].

To our knowledge, the first published results on the numerical solution of Einstein's equations, using spectral methods is the spherically-symmetric collapse of a neutron star to a black hole by Gourgoulhon in 1991 [69]. He used the spectral methods as they have been developed in the Meudon group by Bonazzola and Marck [35]. Later, studies of fast rotating neutron stars [33] (stationary axisymmetric models), the collapse of a neutron star in tensor-scalar theory of gravity [109] (spherically-symmetric dynamical spacetime) and quasi-equilibrium configurations of binary neutron stars [31] and of black holes [81] (three-dimensional and stationary spacetimes) have grown in complexity until the three-dimensional unsteady numerical solution of Einstein's equations [29]. On the other hand, the first fully three-dimensional evolution of the whole Einstein system has been achieved in 2001 by Kidder *et al.* [91], where a single black hole was evolved until  $t \simeq 600M - 1300M$ , using excision techniques. They used spectral methods as developed in the Cornell-Caltech group by Kidder *et al.* [89] and Pfeiffer *et al.* [120]. Since then, they have focused on the evolution of a binary black hole system, which has been evolved until  $t \simeq 600M$  by Scheel *et al.* [129]. Other groups (for instance Ansorg *et al.* [10], Bartnik and Norton [18], Frauendiener [62] and Tichy [148]) have also used spectral methods to solve Einstein's equations; chapters 4 and 5 are devoted to a more detailed review of all these works.

### 1.3 A simple example

Before entering the details of spectral methods in chapters 2 and 3, let us give here their spirit with the simple example of the Poisson equation in a spherical shell:

$$\Delta\phi = \sigma, \tag{7}$$

where  $\Delta$  is the Laplace operator (101) expressed in spherical coordinates  $(r, \theta, \varphi)$  (see also Sec. 3.3). We want to solve Eq. (7) in the domain where  $0 < R_{\min} \leq r \leq R_{\max}$ ,  $\theta \in [0, \pi]$ ,  $\varphi \in [0, 2\pi)$ . This Poisson equation naturally arises in numerical relativity when, for example, solving for initial conditions or the Hamiltonian constraint in the 3+1 formalism [71]: the linear part of these equations can be cast into the form (7), and the non-linearities put into the source  $\sigma$ , with an iterative scheme on  $\phi$ .

First, the angular parts of both fields shall be decomposed onto a (finite) set of spherical harmonics  $\{Y_\ell^m\}$  (see Sec. 3.3.2):

$$\sigma(r, \theta, \varphi) \simeq \sum_{\ell=0}^{\ell_{\max}} \sum_{m=-\ell}^{m=\ell} s_{\ell m}(r) Y_\ell^m(\theta, \varphi), \quad (8)$$

with a similar formula relating  $\phi$  to the radial functions  $f_{\ell m}(r)$ . Because spherical harmonics are eigenfunctions of the angular part of the Laplace operator, the Poisson equation can be equivalently solved as a set of ordinary differential equations for each couple  $(\ell, m)$ , in terms of the coordinate  $r$ :

$$\forall(\ell, m), \quad \frac{d^2 f_{\ell m}}{dr^2} + \frac{2}{r} \frac{df_{\ell m}}{dr} - \frac{\ell(\ell+1)f_{\ell m}}{r^2} = s_{\ell m}(r). \quad (9)$$

We then map

$$\begin{aligned} [R_{\min}, R_{\max}] &\rightarrow [-1, 1] \\ r &\mapsto \xi = \frac{2r - R_{\max} - R_{\min}}{R_{\max} - R_{\min}}, \end{aligned} \quad (10)$$

and decompose each field onto a (finite) base of Chebyshev polynomials  $\{T_i\}_{i=0\dots N}$  (see Sec. 2.4.2):

$$\begin{aligned} s_{\ell m}(\xi) &= \sum_{i=0}^N c_{i\ell m} T_i(\xi), \\ f_{\ell m}(\xi) &= \sum_{i=0}^N a_{i\ell m} T_i(\xi). \end{aligned} \quad (11)$$

Each function  $f_{\ell m}(r)$  can be regarded as a column-vector  $A_{\ell m}$  of its  $N+1$  coefficients  $a_{i\ell m}$  in this base; the linear differential operator on the left-hand side of Eq. (9) being thus a matrix  $L_{\ell m}$  acting on this vector:

$$L_{\ell m} A_{\ell m} = S_{\ell m}, \quad (12)$$

with  $S_{\ell m}$  being the vector of the  $N+1$  coefficients  $c_{i\ell m}$  of  $s_{\ell m}(r)$ . This matrix can be computed from the recurrence relations fulfilled by the Chebyshev polynomials and their derivatives (see Sec. 2.4.2 for details).

The matrix  $L$  is singular, because the problem (7) is ill-posed. One must indeed specify boundary conditions at  $r = R_{\min}$  and  $r = R_{\max}$ . For simplicity, let us suppose

$$\forall(\theta, \varphi), \quad \phi(r = R_{\min}, \theta, \varphi) = \phi(r = R_{\max}, \theta, \varphi) = 0. \quad (13)$$

To impose these boundary conditions, we shall adopt the tau methods (see Sec. 2.5.2): we build the matrix  $\bar{L}$ , taking  $L$  and replacing the last two lines by the boundary conditions, expressed in terms of the coefficients from the properties of Chebyshev polynomials:

$$\forall(\ell, m), \quad \sum_{i=0}^N (-1)^i a_{i\ell m} = \sum_{i=0}^N a_{i\ell m} = 0. \quad (14)$$

Eqs. (14) are equivalent to the boundary conditions (13), within the considered spectral approximation, and they represent the last two lines of  $\bar{L}$ , which can now be inverted and give the coefficients of the solution  $\phi$ .

If one summarizes the steps:

1. Setup an adapted grid for the computation of spectral coefficients (*e.g.* equidistant in the angular directions and Chebyshev-Gauss-Lobatto collocation points, see Sec. 2.4.2);
2. Get the values of the source  $\sigma$  on these grid points;
3. Perform a spherical-harmonics transform (for example using some available library [106]), followed by the Chebyshev transform (using a Fast Fourier Transform-FFT, or a Gauss-Lobatto quadrature) of the source  $\sigma$ ;
4. For each couple of values  $(\ell, m)$ , build the corresponding matrix  $\bar{L}$ , with the boundary conditions and invert the system (using any available linear-algebra package) with the coefficients of  $\sigma$  as the right-hand side;
5. Perform the inverse spectral transform to get the values of  $\phi$  on the grid points, from its coefficients.

As shown by Grandclément *et al.* [82], or in Sec. 2.5.2 but for a different differential equation, the error with this technique would decay as  $e^{-\ell_{\max}} \cdot e^{-N}$ , provided that the source sigma is smooth. Machine round-off accuracy can be reached with  $\ell_{\max} \sim N \sim 30$ , which makes the matrix inversions of step 4 very cheap in terms of CPU, and the whole method in terms of memory usage too. These are the main advantages of using spectral methods, as it shall be shown in the following sections.

## 2 Theoretical Foundations

In this section the mathematical basis of spectral methods are presented. Some generalities about approximating functions with polynomials are first given. The basic formulae of spectral approximation are then given and two sets of polynomials are discussed (Legendre and Chebyshev polynomials). A particular emphasize is put on convergence properties (i.e. the way the spectral approximate converges to the real function).

In Sec. 2.5, three different methods for solving an ordinary differential equation are exhibited and applied on a simple problem. Sec. 2.6 is concerned with multi-domain techniques. After giving some motivations for the use of multi-domain decomposition, three different implementations are discussed and their merits discussed. One simple example is given, which uses only two domains.

Let us mention that this section is only concerned with 1-dimensional problems (see Sec. 3 for problems in higher dimensions).

### 2.1 Best polynomial approximation

Polynomials are the only functions that a computer can exactly evaluate and so it is natural to try to approximate any function by a polynomial. When considering spectral methods, one will use high-order polynomials on a few domains. This is to be contrasted with finite difference schemes, for instance, where only local polynomials of low degree are considered.

In this particular section, real functions of  $[-1, 1]$  are considered. A theorem due to Weierstrass, 1885, states that the set  $\mathbb{P}$  of all polynomials is a dense subspace of all the continuous functions on  $[-1, 1]$ , with the norm  $\|\cdot\|_\infty$ . This maximum norm is defined as

$$\|f\|_\infty = \max_{x \in [-1, 1]} |f(x)|. \quad (15)$$

This means that, for any continuous function  $f$  of  $[-1, 1]$ , there exists a sequence of polynomials  $(p_n), n \in \mathbb{N}$  that converges *uniformly* towards  $f$  :

$$\lim_{n \rightarrow \infty} \|f - p_n\|_\infty = 0. \quad (16)$$

This theorem shows that it is probably a good idea to approximate continuous functions by polynomials.

Given a continuous function  $f$ , the best polynomial approximation of degree  $N$ , is the polynomial  $p_N^*$  that minimizes the norm of the difference between  $f$  and itself:

$$\|f - p^*\|_\infty = \min \{\|f - p\|_\infty, p \in \mathbb{P}_N\}. \quad (17)$$

*Chebyshev alternate theorem* states that for any continuous function  $f$ ,  $p_N^*$  is unique. There exist  $N+2$  points  $x_i \in [-1, 1]$  such that the error is exactly attained at those points, in an alternate manner :

$$f(x_i) - p_N^*(x_i) = (-1)^{i+\delta} \|f - p_N^*\|_\infty, \quad (18)$$

where  $\delta = 0$  or  $\delta = 1$ . An example of a function and its best polynomial approximation is shown of Fig. 1.

### 2.2 Interpolation on a Grid

A *grid*  $X$  on the interval  $[-1, 1]$  is a set of  $N+1$  points  $x_i \in [-1, 1], 0 \leq i \leq N$ . Those points are called the *nodes* of the grid  $X$ .

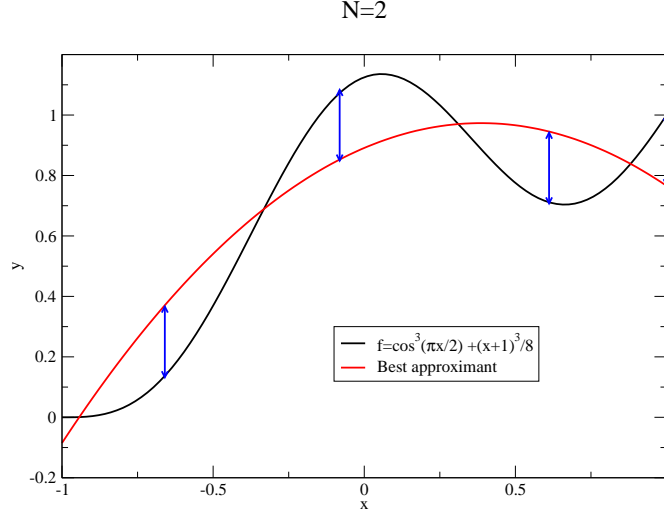


Figure 1: Function  $f = \cos^3(\pi x/2) + (x+1)^3/8$  (black curve) and its best approximation of degree 2 (red curve). The blue arrows denote the 4 points where the maximum error is attained.

Let us consider a continuous function  $f$  and a grid  $X$  with  $N + 1$  nodes  $x_i$ . Then, there exist a unique polynomial of degree  $N$ ,  $I_N^X f$ , that coincides with  $f$  at each node :

$$I_N^X f(x_i) = f(x_i) \quad 0 \leq i \leq N. \quad (19)$$

$I_N^X f$  is called the interpolant of  $f$  through the grid  $X$ .  $I_N^X f$  can be expressed in terms of the Lagrange cardinal polynomials:

$$I_N^X f = \sum_{i=0}^N f(x_i) l_i^X(x), \quad (20)$$

where the  $l_i^X$  are the Lagrange cardinal polynomials. By definition,  $l_i^X$  is the unique polynomial of degree  $N$ , that vanishes at all nodes of the grid  $X$  but at  $x_i$ , where it is 1. It is easy to show that the Lagrange cardinal polynomials can be written as

$$l_i^X(x) = \prod_{j=0, j \neq i}^N \frac{x - x_j}{x_i - x_j}. \quad (21)$$

Figure 2 shows some examples of Lagrange cardinal polynomials and an example of a function and its interpolant on a uniform grid can be seen on Fig. 3.

Thanks to Chebyshev alternate theorem, one can see that the best approximation of degree  $N$  is an interpolant of the function at  $N + 1$  nodes. However, in general, the associated grid is not known. The difference between the error made by interpolating on a given grid  $X$  can be compared to the smallest possible error for the best approximation. One can show that :

$$\|f - I_N^X f\|_\infty \leq (1 + \Lambda_N(X)) \|f - p^*\|_\infty, \quad (22)$$



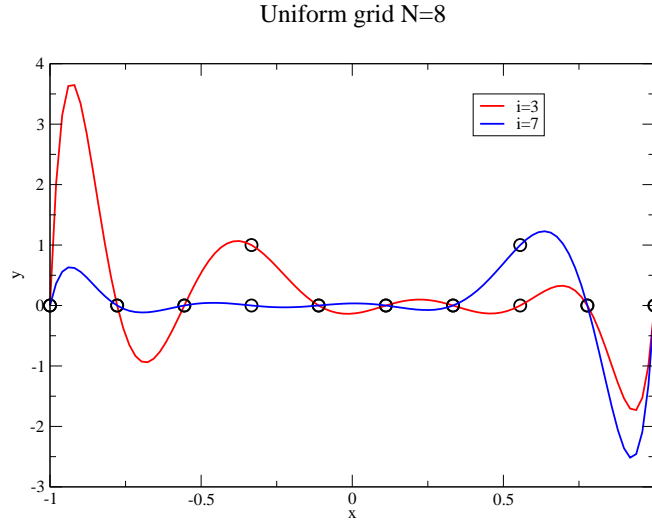


Figure 2: *Lagrange cardinal polynomials  $l_3^X$  (red curve) and  $l_7^X$  on an uniform grid with  $N = 8$ . The black circles denote the nodes of the grid.*

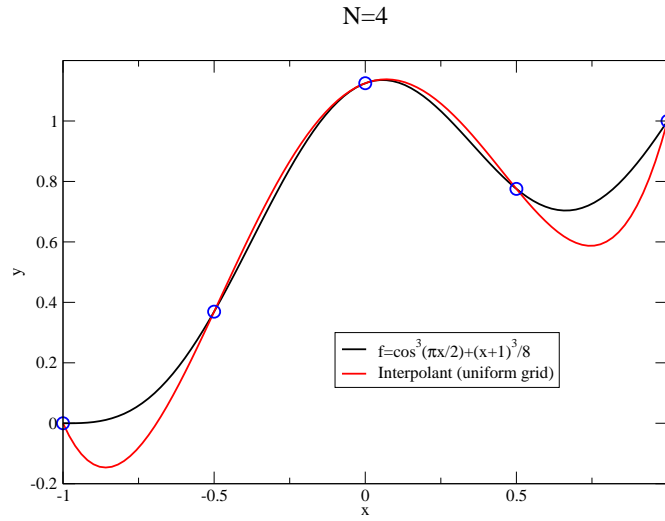


Figure 3: *Function  $f = \cos^3(\pi x/2) + (x+1)^3/8$  (black curve) and its interpolant (red curve) on a uniform grid of 5 nodes. The blue circles show the position of the nodes.*

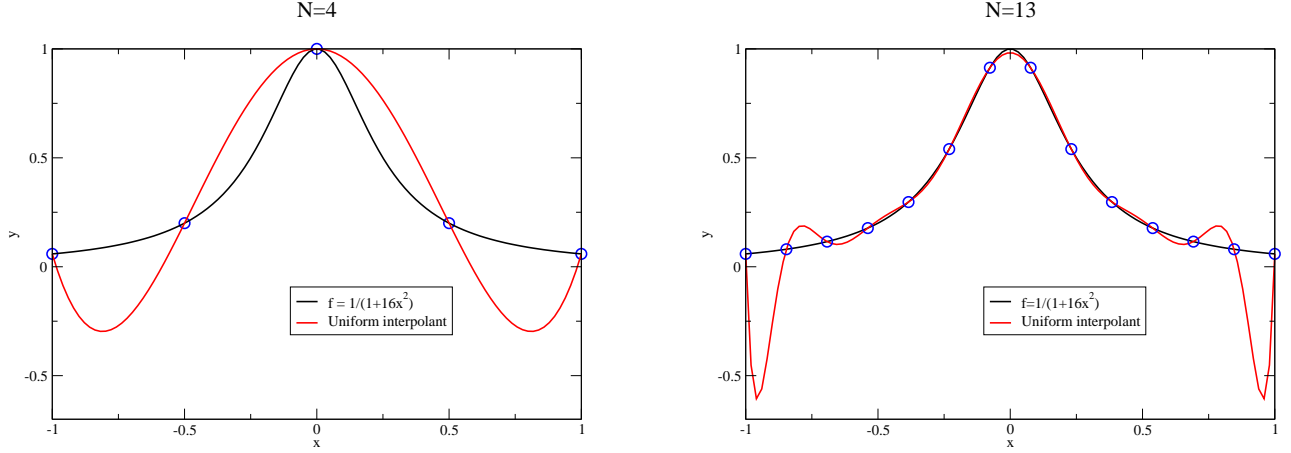


Figure 4: Function  $f = \frac{1}{1+16x^2}$  (black curve) and its interpolant (red curve) on a uniform grid of 5 nodes (left panel) and 14 nodes (right panel). The blue circles show the position of the nodes.

where  $\Lambda$  is the *Lebesgue constant* of the grid  $X$  and is defined as :

$$\Lambda_N(X) = \max_{x \in [-1, 1]} \sum_{i=0}^N |l_i^x(x)|. \quad (23)$$

A theorem by Erdős (1961) states that, for any choice of grid  $X$ , there exist a constant  $C > 0$  such that :

$$\Lambda_N(X) > \frac{2}{\pi} \ln(N+1) - C. \quad (24)$$

It immediately follows that  $\Lambda(N) \rightarrow \infty$  when  $N \rightarrow \infty$ . This implies that for any grid, there always exists at least one continuous function  $f$  which interpolant does not converge uniformly to  $f$ . An example of such failure of the convergence is shown on Fig. 4, where the interpolant of the function  $f = 1/(1+16x^2)$  is clearly not uniform (see the behavior near the boundaries of the interval). This is known as the Runge phenomenon.

Moreover, a theorem by Cauchy states that, for all functions  $f \in \mathcal{C}^{(N+1)}$ , the interpolation error, on a grid  $X$  of  $N+1$  nodes is given by

$$f(x) - I_N^X(x) = \frac{f^{(N+1)}(\epsilon)}{(N+1)!} w_{N+1}^X(x), \quad (25)$$

where  $\epsilon \in [-1, 1]$ .  $w_{N+1}^X$  is the nodal polynomial of  $X$ , being the only polynomial of degree  $N+1$ , with a leading coefficient 1 and that vanishes on the nodes of  $X$ . It is then easy to show that

$$w_{N+1}^X(x) = \prod_{i=0}^N (x - x_i). \quad (26)$$

On equation (25), one has a priori no control on the term involving  $f^{(N+1)}$ . For a given function, this can be rather large and this is indeed the case for the function  $f$  shown on Fig. 4. However, one can hope to minimize the interpolation error by choosing a grid such that the nodal polynomial

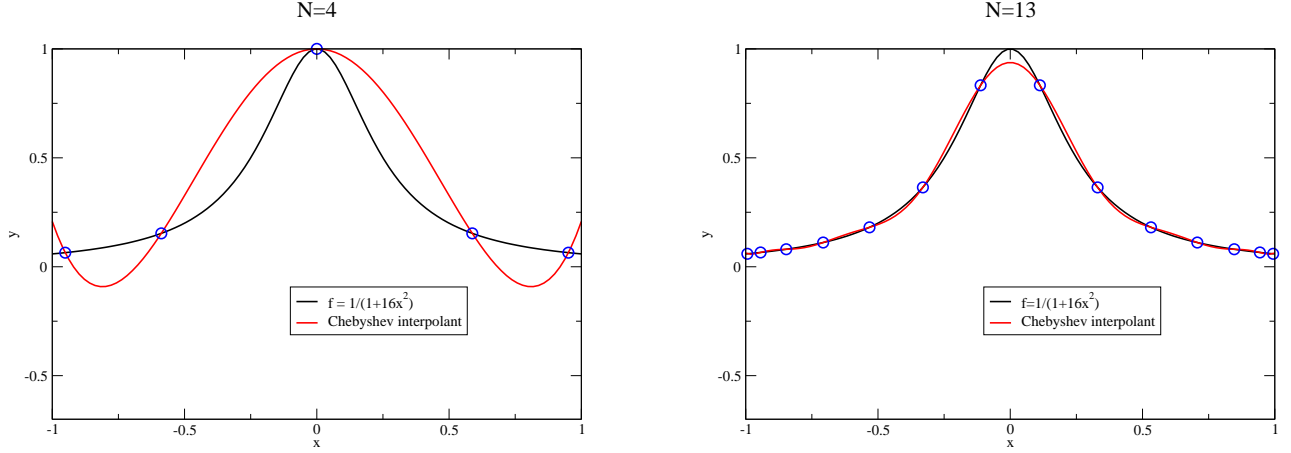


Figure 5: Same thing as Fig. 4 but using a grid based on the zeros of Chebyshev polynomials. The Runge phenomenon is no longer present.

is as small as possible. A theorem by Chebyshev states that this choice is unique and is given by a grid which nodes are the zeros of the Chebyshev polynomial  $T_{N+1}$  (see Sec. 2.3 for more details on Chebyshev polynomials). With such a grid, one can achieve

$$\|w_{N+1}^X\|_{\infty} = \frac{1}{2^N}, \quad (27)$$

which is the smallest possible value. So, a grid based on nodes of Chebyshev polynomials can be expected to perform better than a standard uniform one. This is what can be seen on Fig. 5, which shows the same thing than Fig. 4 but with a Chebyshev grid. Clearly, the Runge phenomenon is no longer present. It can be checked, that, for this choice of function  $f$ , the uniform convergence of the interpolant to the function is recovered.

## 2.3 Polynomial Interpolation

### 2.3.1 Orthogonal polynomials

Spectral methods are based on the notion of *orthogonal polynomials*. In order to define orthogonality, one has to define the scalar product of two functions, on an interval  $[-1, 1]$ . Let us consider a positive function  $w$  of  $[-1, 1]$  called the *measure*. The scalar product of  $f$  and  $g$ , with respect to this measure is defined as :

$$(f, g)_w = \int_{x \in [-1, 1]} f(x) g(x) w(x) dx. \quad (28)$$

A basis of  $P_{\mathbb{N}}$  is then a set of  $N + 1$  polynomials  $p_n$ , each of degree  $n$  that are orthogonal :  $(p_i, p_j)_w = 0$  for  $i \neq j$ .

The projection  $P_N f$  of a function  $f$  on this basis is then

$$P_N f = \sum_{n=0}^N \hat{f}_n p_n, \quad (29)$$

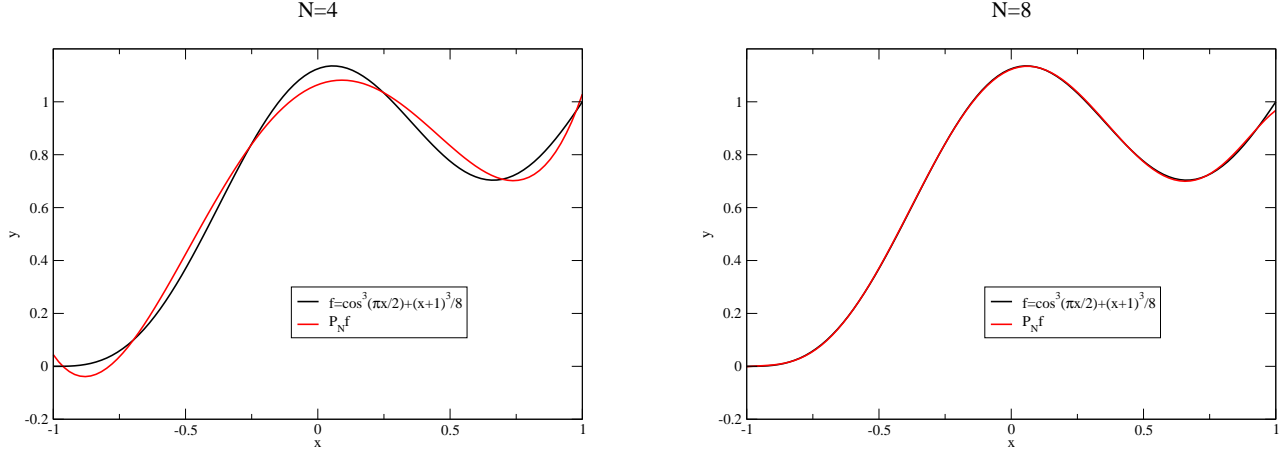


Figure 6: Function  $f = \cos^3(\pi x/2) + (x+1)^3/8$  (black curve) and its projection on Chebyshev polynomials (red curve), for  $N = 4$  (left panel) and  $N = 8$  (right panel).

where the coefficients of the projection are given by

$$\hat{f}_n = \frac{(f, p_n)}{(p_n, p_n)}. \quad (30)$$

The difference between  $f$  and its projection goes to zero when  $N$  increases :

$$\|f - P_N f\|_\infty \rightarrow 0 \quad \text{when} \quad N \rightarrow \infty. \quad (31)$$

Figure 6 shows the function  $f = \cos^3(\pi x/2) + (x+1)^3/8$  and its projection on Chebyshev polynomials (see Sec. 2.4.2), for  $N = 4$  and  $N = 8$ , illustrating the rapid convergence of  $P_N f$  to  $f$ .

At first sight, the projection seems to be an interesting mean of numerically representing a function. However, in practice, this is not the case. Indeed, to determine the projection of a function, one needs to compute the integrals (30), which requires the evaluation of  $f$  at a great number of points, thus making the all numerical scheme impracticable.

### 2.3.2 Gaussian quadratures

The main theorem of Gaussian quadratures states that, given a measure  $w$ , there exist  $N + 1$  positive reals  $w_n$  and  $N + 1$  reals  $x_n \in [-1, 1]$  such that:

$$\forall f \in \mathbb{P}_{2N+\delta}, \quad \int_{[-1,1]} f(x) w(x) dx = \sum_{n=0}^N f(x_n) w_n. \quad (32)$$

The  $w_n$  are called the *weights* and the  $x_n$  are the collocation points. The degree of applicability of the theorem depends on the integer  $\delta$ , which can take the following values.

- Gauss quadrature :  $\delta = 1$ .
- Gauss-Radau :  $\delta = 0$  and  $x_0 = -1$ .

- Gauss-Lobatto :  $\delta = -1$  and  $x_0 = -1$  and  $x_N = 1$ .

Gauss quadrature is more appealing because it applies to polynomials of higher degree but Gauss-Lobatto quadrature is often more useful for numerical purposes because the outermost collocation points coincide with the boundaries of the interval making it easier to impose continuities or boundary conditions.

### 2.3.3 Spectral interpolation

As already stated in 2.3.1, the main shortcoming of projecting a function on orthogonal polynomials comes from the difficulty to compute the integrals (30). The idea of spectral methods is to approximate the coefficients of the projection by making use of the Gaussian quadratures. By doing so, one can define the *interpolant* of a function  $f$  by

$$I_N f = \sum_{n=0}^N \tilde{f}_n p_n(x), \quad (33)$$

where

$$\tilde{f}_n = \frac{1}{\gamma_n} \sum_{i=0}^N f(x_i) p_n(x_i) w_i \quad \text{and} \quad \gamma_n = \sum_{i=0}^N p_n^2(x_i) w_i. \quad (34)$$

The  $\tilde{f}_n$  exactly coincides with the coefficients  $\hat{f}_n$ , if the Gaussian quadrature is applicable for computing (30), that is for all  $f \in \mathbb{P}_{N+\delta}$ . So, in general,  $I_N f \neq P_N f$  and the difference between the two is called the *aliasing error*. The advantage of using the  $\tilde{f}$  is that they are computed by estimating  $f$  at the  $N + 1$  collocation points only.

One can show that  $I_N f$  and  $f$  coincide at the collocation points :  $I_N f(x_i) = f(x_i)$  so that  $I_N$  interpolates  $f$  on the grid which nodes are the collocation points. Figure 7 shows the function  $f = \cos^3(\pi x/2) + (x+1)^3/8$  and its spectral interpolation using Chebyshev polynomials, for  $N = 4$  and  $N = 6$ .

### 2.3.4 Two equivalent descriptions

The description of a function  $f$  in terms of its spectral interpolation can be given in two different but equivalent spaces:

- in the configuration space if the function is described by its value at the  $N + 1$  collocation points  $f(x_i)$ .
- in the coefficient space if one works with the  $N + 1$  coefficients  $\tilde{f}_i$ .

There is a bijection between the two spaces and the following relations enable to go from one description to the other:

- the coefficients can be computed from the values of  $f(x_i)$  using Eq. (34).
- the values at the collocation points are expressed in terms of the coefficients by making use of the definition of the interpolant (33):

$$f(x_i) = \sum_{n=0}^N \tilde{f}_n p_n(x_i). \quad (35)$$

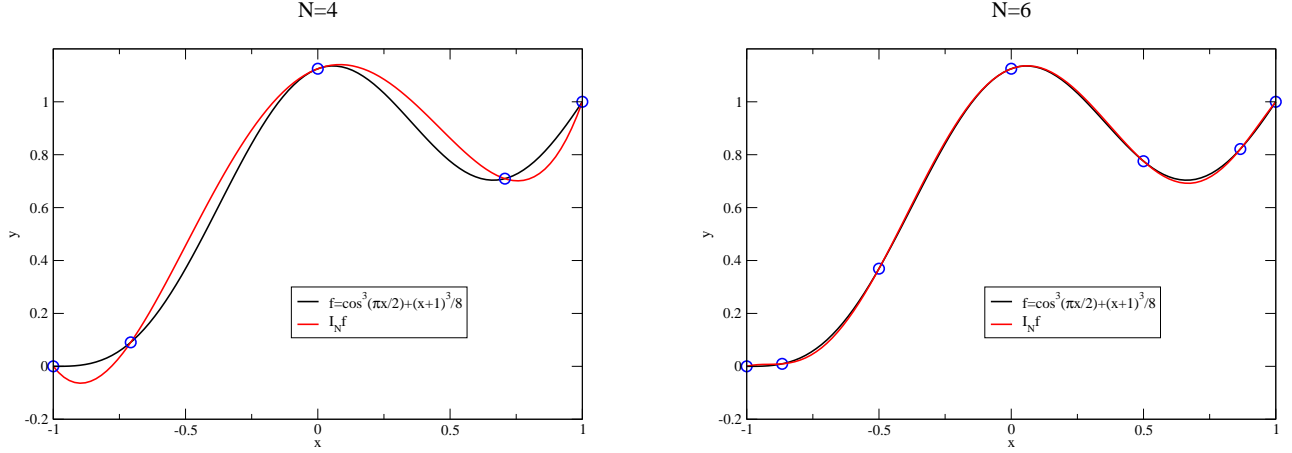


Figure 7: Function  $f = \cos^3(\pi x/2) + (x+1)^3/8$  (black curve) and its interpolant  $I_N f$  Chebyshev polynomials (red curve), for  $N = 4$  (left panel) and  $N = 6$  (right panel). The collocation points are denoted by the blue circles and correspond to Gauss-Lobatto quadrature.

Depending on the operation one has to perform on a given function, it may be more clever to work in one space or the other. For instance, the square root of a function is very easily given in the collocation space by  $\sqrt{f}(x_i)$ , whereas the derivative can be computed in the coefficient space, if, and this is generally the case, the derivatives of the basis polynomials are known, by

$$f' = \sum_{n=0}^N \tilde{f}_n p'_n(x).$$

## 2.4 Usual polynomials

In this section, some commonly used sets of orthogonal functions are presented.

### 2.4.1 Legendre polynomials

Legendre polynomials  $P_n$  are orthogonal on  $[-1, 1]$  with respect to the measure  $w(x) = 1$ . Moreover, the scalar product of two polynomials is given by :

$$(P_n, P_m) = \int_{-1}^1 P_n P_m dx = \frac{2}{2n+1} \delta_{mn}. \quad (36)$$

Starting from  $P_0 = 1$  and  $P_1 = x$ , the successive polynomials can be computed by the following recurrence expression:

$$(n+1) P_{n+1}(x) = (2n+1)x P_n(x) - n P_{n-1}(x). \quad (37)$$

Amongst the various properties of Legendre polynomials, one can note that i)  $P_n$  has the same parity as  $n$ . ii)  $P_n$  is of degree  $n$ . iii)  $P_n(\pm 1) = (-1)^n$ . iv)  $P_n$  has exactly  $n$  zeros on  $[-1, 1]$ . The first polynomials are shown on Fig. 8.

The weights and location of the collocation points associated with Legendre polynomials depend on the choice of quadrature.

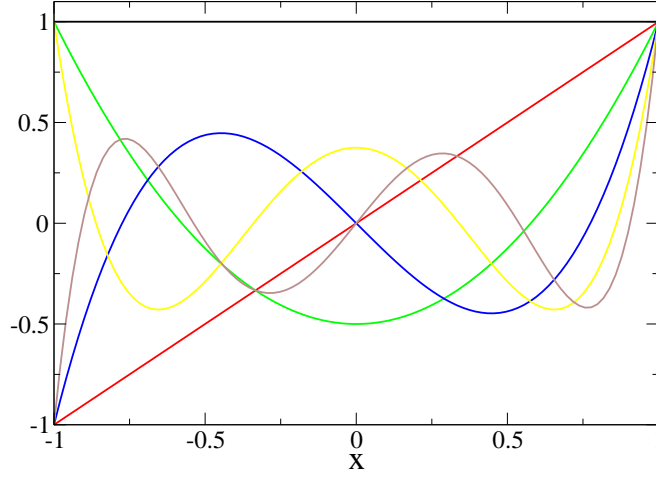


Figure 8: *First Legendre polynomials, from  $P_0$  to  $P_5$ .*

- Legendre-Gauss :  $x_i$  are the nodes of  $P_{N+1}$  and  $w_i = \frac{2}{(1-x_i^2) [P'_{N+1}(x_i)]^2}$ .
- Legendre-Gauss-Radau :  $x_0 = -1$  and the  $x_i$  are the nodes of  $P_N + P_{N+1}$ . The weights are given by  $w_0 = \frac{2}{(N+1)^2}$  and  $w_i = \frac{1}{(N+1)^2}$ .
- Legendre-Gauss-Lobatto :  $x_0 = -1$ ,  $x_N = 1$  and  $x_i$  are the nodes of  $P'_N$ . The weights are  $w_i = \frac{2}{N(N+1)} \frac{1}{[P'_N(x_i)]^2}$ .

Those values are not analytic but can be computed numerically in an efficient way.

Some elementary operations can be easily performed on the coefficient space. Let us assume that a function  $f$  is given by its coefficients  $a_n$  so that  $f = \sum_{n=0}^N a_n P_n$ . Then the coefficients  $b_n$  of  $Hf = \sum_{n=0}^N b_n P_n$  can be found as a function of the  $a_n$ , for various operators  $H$ . For instance,

- if  $H$  is the multiplication by  $x$  then :

$$b_n = \frac{n}{2n-1} a_{n-1} + \frac{n+1}{2n+3} a_{n+1} \quad (n \geq 1). \quad (38)$$

- if  $H$  is the derivation :

$$b_n = (2n+1) \sum_{p=n+1, p+n \text{ odd}}^N a_p. \quad (39)$$

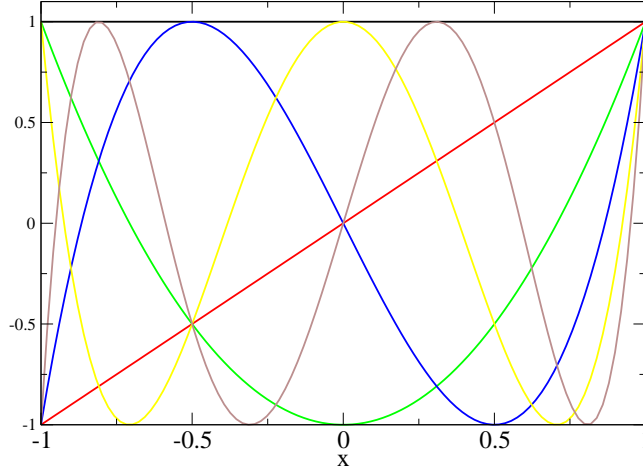


Figure 9: *First Chebyshev polynomials, from  $T_0$  to  $T_5$ .*

- if  $H$  is the second derivation :

$$b_n = (n + 1/2) \sum_{p=n+2, p+n \text{ even}}^N [p(p+1) - n(n+1)] a_p. \quad (40)$$

Those kind of relations enable to represent the action of  $H$  as a matrix acting on the vector of the  $a_n$ , the product being the coefficients of  $Hf$ , i.e. the  $b_n$ .

#### 2.4.2 Chebyshev polynomials

Chebyshev polynomials, denoted by  $T_n$ , are orthogonal on  $[-1, 1]$  with respect to the measure  $w = 1/\sqrt{1-x^2}$  and the scalar product of two polynomials is

$$(T_n, T_m) = \int_{-1}^1 \frac{T_n T_m}{\sqrt{1-x^2}} dx = \frac{\pi}{2} (1 + \delta_{0n}) \delta_{mn}. \quad (41)$$

Given that  $T_0 = 1$  and  $T_1 = x$ , the higher order polynomials can be obtained by making use of the recurrence

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \quad (42)$$

This implies the following simple properties. i)  $T_n$  has the same parity as  $n$ . ii)  $T_n$  is of degree  $n$ . iii)  $T_n(\pm 1) = (-1)^n$ . iv)  $T_n$  has exactly  $n$  zeros on  $[-1, 1]$ . The first polynomials are shown on Fig. 9.

Contrary to the Legendre case, both the weights and position of the collocation points are analytical and given by :

- Chebyshev-Gauss :  $x_i = \cos \frac{(2i+1)\pi}{2N+2}$  and  $w_i = \frac{\pi}{N+1}$ .



- Chebyshev-Gauss-Radau :  $x_i = \cos \frac{2\pi i}{2N+1}$ . The weights are  $w_0 = \frac{\pi}{2N+1}$  and  $w_i = \frac{2\pi}{2N+1}$
- Chebyshev-Gauss-Lobatto :  $x_i = \cos \frac{\pi i}{N}$ . The weights are  $w_0 = w_N = \frac{\pi}{2N}$  and  $w_i = \frac{\pi}{N}$ .

As for the Legendre case, the action of various linear operators  $H$  can be expressed in the coefficient space. This means that the coefficients  $b_n$  of  $Hf$  are given as functions of the coefficients  $a_n$  of  $f$ . For instance,

- if  $H$  is the multiplication by  $x$  then :

$$b_n = \frac{1}{2} [(1 + \delta_{0n-1}) a_{n-1} + a_{n+1}] \quad (n \geq 1). \quad (43)$$

- if  $H$  is the derivation :

$$b_n = \frac{2}{(1 + \delta_{0n})} \sum_{p=n+1, p+n \text{ odd}}^N p a_p. \quad (44)$$

- if  $H$  is the second derivation :

$$b_n = \frac{1}{(1 + \delta_{0n})} \sum_{p=n+2, p+n \text{ even}}^N p(p^2 - n^2) a_p. \quad (45)$$

### 2.4.3 Convergence properties

One of the main advantage of spectral method is the very fast convergence of the interpolant  $I_N f$  to the function  $f$ , at least for smooth enough functions. Let us consider a  $\mathcal{C}^m$  function  $u$ , then, one can place the following upper bounds on the difference between  $u$  and its interpolant  $I_N u$  :

- For Legendre :

$$\|I_N u - u\|_{L^2} \leq \frac{C_1}{N^{m-1/2}} \sum_{k=0}^m \|u^{(k)}\|_{L^2}. \quad (46)$$

- For Chebyshev :

$$\|I_N u - u\|_{L_w^2} \leq \frac{C_2}{N^m} \sum_{k=0}^m \|u^{(k)}\|_{L_w^2}. \quad (47)$$

$$\|I_N u - u\|_{\infty} \leq \frac{C_3}{N^{m-1/2}} \sum_{k=0}^m \|u^{(k)}\|_{L_w^2}. \quad (48)$$

The  $C_i$  are some positive constants. An interesting limit of the above estimates concerns a  $\mathcal{C}^\infty$  function. One can then see that the difference between  $u$  and  $I_N u$  decays faster than any power of  $N$ . It implies that the error decays exponentially and one talks about an evanescent error. An example of this very fast convergence is shown on Fig. 10. The error clearly decays as an exponential, until the level of  $10^{-14}$  of the precision of the computation is reached (one is working in double precision in this particular case). Fig. 10 illustrates the fact that, with spectral methods, very good accuracy can be reached with only a moderate number of coefficients.

If the function is less regular (i.e. not  $\mathcal{C}^\infty$ ), the error is no longer exponential and only decays as a power-law, thus making the use of spectral method less appealing. This effect is called the Gibbs phenomenon. It can be easily seen on the worst possible case, the one of a discontinuous function ( $m = 0$ ). In the case, the estimates (46-48) do not even ensure convergence at all. On Fig. 11 one shows a step function and its interpolant, for various values of  $N$ . One can see that the maximum difference between the function and its interpolant is not even going to zero when  $N$  is increased, in agreement with the application of (48) for  $m = 0$ .

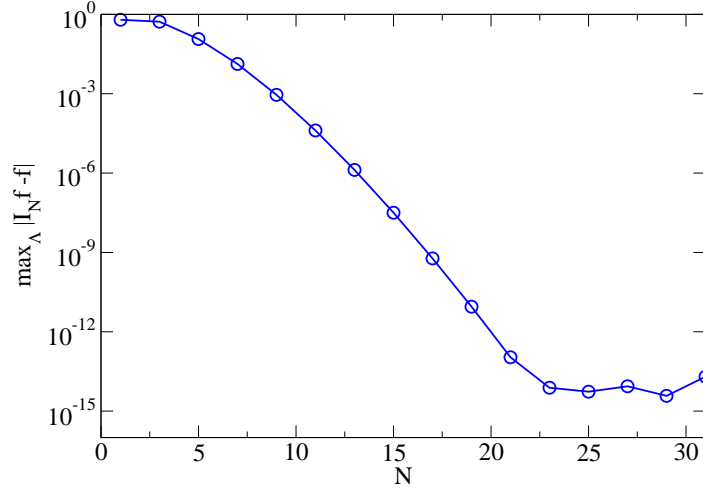


Figure 10: Maximum difference between  $f = \cos^3(\pi x/2) + (x+1)^3/8$  and its interpolant  $I_N f$ , as a function of  $N$ .

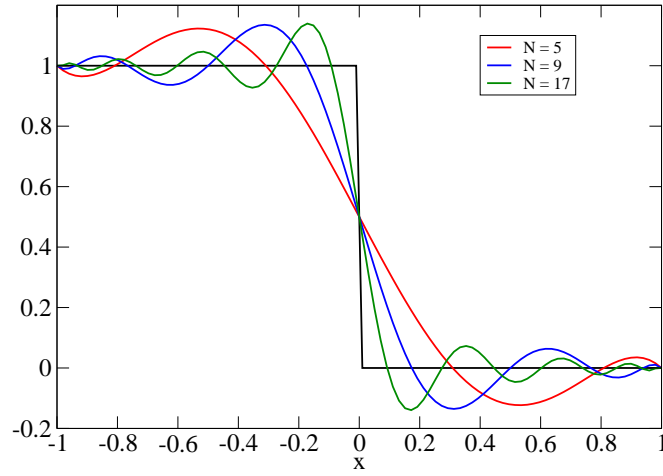


Figure 11: Step function (black curve) and its interpolant, for various values of  $N$ .

#### 2.4.4 Trigonometrical functions

A detailed presentation of the theory of Fourier transform is beyond the scope of this work. However, there is a close link between the so-called *discrete Fourier transform* and the spectral interpolation and this is briefly outlined here.

The Fourier transform  $Pf$  of a function  $f$  of  $[0, 2\pi]$  is given by :

$$Pf(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx). \quad (49)$$

The Fourier transform is known to converge rather rapidly to the function itself. However, the coefficients  $a_n$  and  $b_n$  are given by integrals like  $\int_0^{2\pi} f(x) \cos(nx) dx$  that can not be easily computed (as it was the case for the projection of a function on orthogonal polynomials in Sec. 2.3.1).

The solution to this problem is also very similar to the use of the Gaussian quadratures. Let us introduce  $N + 1$  collocation points  $x_i = 2\pi i/(N + 1)$ . Then, the *discrete Fourier coefficients* with respect to those points are :

$$\tilde{a}_0 = \frac{1}{N} \sum_{k=1}^N f(x_k) \quad (50)$$

$$\tilde{a}_n = \frac{2}{N} \sum_{k=1}^N f(x_k) \cos(nx_k) \quad (51)$$

$$\tilde{b}_n = \frac{2}{N} \sum_{k=1}^N f(x_k) \sin(nx_k) \quad (52)$$

and the interpolant  $I_N f$  is then given by :

$$I_N f(x) = \tilde{a}_0 + \sum_{n=1}^N \tilde{a}_n \cos(nx) + \sum_{n=1}^N \tilde{b}_n \sin(nx). \quad (53)$$

The approximation made by using the discrete coefficients in place of the real ones is of the same nature than the one made when computing the coefficients of the projection (30) by means of the Gaussian quadratures. Let us mention that, in the case of a discrete Fourier transform, the first and last collocation points lies on the boundary of the interval, as for a Gauss-Lobatto quadrature. As for the polynomial interpolation, the convergence of  $I_N f$  to  $f$  is exponential, for all periodic and  $C^\infty$  functions.

#### 2.4.5 Choice of basis

For periodic functions of  $[0, 2\pi]$ , the discrete Fourier transform is the natural choice of basis. If the considered function has also some symmetries, one can use a subset of the trigonometrical polynomials. For instance, if the function is i) periodic on  $[0, 2\pi]$  and is also odd with respect to  $x = \pi$ , then it can be expanded on sines only.

If the function is not periodic, then it is natural to expand it either on Chebyshev or Legendre polynomials. Chebyshev polynomials have two main advantages. First the associated weights and collocation points are analytical and second, the coefficients can be computed by means of FFT algorithms. The use of an FFT reduces the number of operations from  $N^2$  when using the standard formula Eq. (34) to only  $N \ln N$  operations. For codes where most of the computational time is spent going from one representation space to the other, this may be an interesting feature. The

main advantage of Legendre polynomials is the fact that the associated measure is very simple  $w(x) = 1$ . The multi-domain technique presented in Sec. 2.6.4 is one particular example where such property is required.

## 2.5 Spectral Methods for ODEs

### 2.5.1 Weighted residual method

Let us consider a differential equation of the following form

$$Lu(x) = S(x), \quad x \in [-1, 1], \quad (54)$$

where  $L$  is a linear, second order, differential operator. The problem admits a unique solution once some boundary conditions are prescribed at  $x = 1$  and  $x = -1$ . Typically, one can specify i) the value of  $u$  (Dirichlet-type) ii) the value of its derivative  $\partial_x u$  (Neumann-type) iii) a linear combination of the two (Robin-type).

As for the elementary operations presented in Sec. 2.4.1 and 2.4.2, the action of  $L$  on  $u$  can be expressed by a matrix  $L_{ij}$ . If the coefficients of  $u$  with respect to a given basis are the  $\tilde{u}_i$ , then the coefficients of  $Lu$  are

$$\sum_{j=0}^N L_{ij} \tilde{u}_j. \quad (55)$$

The  $L_{ij}$  can usually be easily computed by combining the action of elementary operations like the second derivation, the first derivation, the multiplication or division by  $x$  (see Sec. 2.4.1 and 2.4.2 for some examples).

A function  $u$  is an admissible solution of the problem if and only if i) it fulfills the boundary conditions exactly (up to machine accuracy) ii) it makes the *residual*  $R = Lu - S$  small. In the weighted residual method, one considers a set of  $N + 1$  test functions  $\xi_n$  on  $[-1, 1]$ . The smallness of  $R$  is enforced by demanding that

$$(R, \xi_k) = 0, \quad \forall k \leq N. \quad (56)$$

As  $N$  increases, the obtained solution is closer and closer to the real one. Depending on the choice of the test functions and the way the boundary conditions are enforced, one gets various solvers. Three classical examples are presented next.

### 2.5.2 The Tau-method

In this particular method, the test functions coincide with the basis used for the spectral expansion, for instance the Chebyshev polynomials. Let us denote  $\tilde{u}_i$  and  $\tilde{s}_i$  the coefficients of the solution  $u$  and the source  $S$  with respect to Chebyshev polynomials.

Given the expression of  $Lu$  in the coefficient space (55) and the fact that the basis polynomials are orthogonal, the residual equations (56) are expressed as

$$\sum_{i=0}^N L_{ni} \tilde{u}_i = \tilde{s}_n, \quad \forall n \leq N, \quad (57)$$

the unknowns being the  $\tilde{u}_i$ . However, as such, this system does not admit a unique solution, due to the homogeneous solutions of  $L$  (i.e. the matrix associated with  $L$  is not invertible) and one has to impose the boundary conditions. In the Tau-method, this is done by relaxing the *last two* equations (57) (i.e. for  $n = N - 1$  and  $n = N$ ) and by replacing them by the boundary conditions at  $x = -1$  and  $x = 1$ .

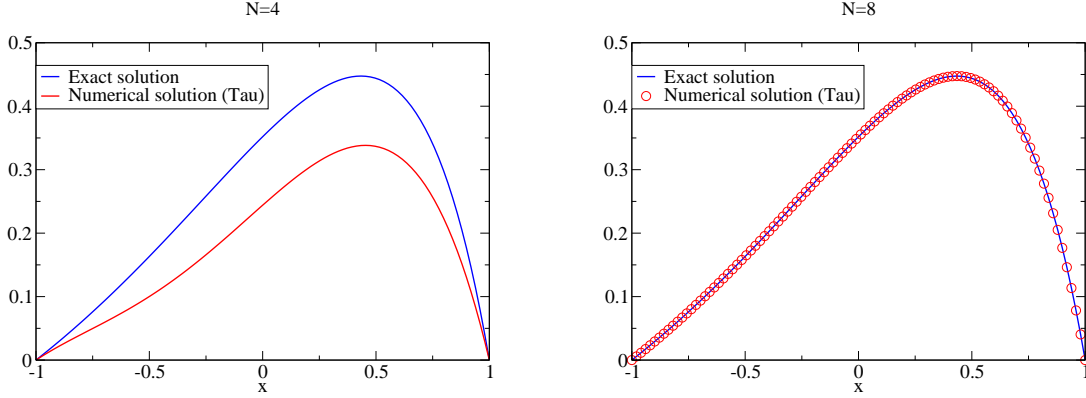


Figure 12: *Exact solution (60) of Eq. (58) (blue curve) and the numerical solution (red curves) computed by means of a Tau-method, for  $N = 4$  (left panel) and  $N = 8$  (right panel).*

The Tau-method thus ensures that  $Lu$  and  $S$  have the same coefficients but for the last ones. If the functions are smooth, then their coefficients should decrease exponentially (evanescent error) and so the “forgotten” conditions are less and less stringent when  $N$  increases, ensuring that the computed solution converges rapidly to the real one.

As an illustration, let us consider the following equation :

$$\frac{d^2u}{dx^2} - 4\frac{du}{dx} + 4u = \exp(x) - \frac{4e}{(1+x^2)} \quad (58)$$

with the following boundary conditions

$$u(x = -1) = 0 \quad \text{and} \quad u(x = 1) = 0. \quad (59)$$

The solution exact is analytical and is

$$u(x) = \exp(x) - \frac{\sinh(1)}{\sinh(2)} \exp(2x) - \frac{e}{(1+x^2)}. \quad (60)$$

Fig. 12 shows the exact solution and the numerical one, for two different values of  $N$ . One can note that the numerical solution converges rapidly to the numerical one, the two being almost indistinguishable for  $N$  as small as  $N = 8$ . The numerical solution exactly fulfills the boundary conditions, no matter what  $N$  is.

### 2.5.3 The collocation method

The collocation method is very similar to the Tau-method. They only differ from the choice of test functions. Indeed, in the collocation method one uses continuous function that are zero at each but one collocation point. They are indeed the Lagrange cardinal polynomials already seen in Sec. 2.2 and can be written as  $\xi_i(x_j) = \delta_{ij}$ . With such test functions, the residual equations (56) are

$$Lu(x_n) = S(x_n), \quad \forall n \leq N. \quad (61)$$

The value of  $Lu$  at each collocation points is easily expressed in terms of the  $\tilde{u}$  by making use of (55) and one gets :

$$\sum_{i=0}^N \sum_{j=0}^N L_{ij} \tilde{u}_j T_i(x_n) = S(x_n), \quad \forall n \leq N. \quad (62)$$

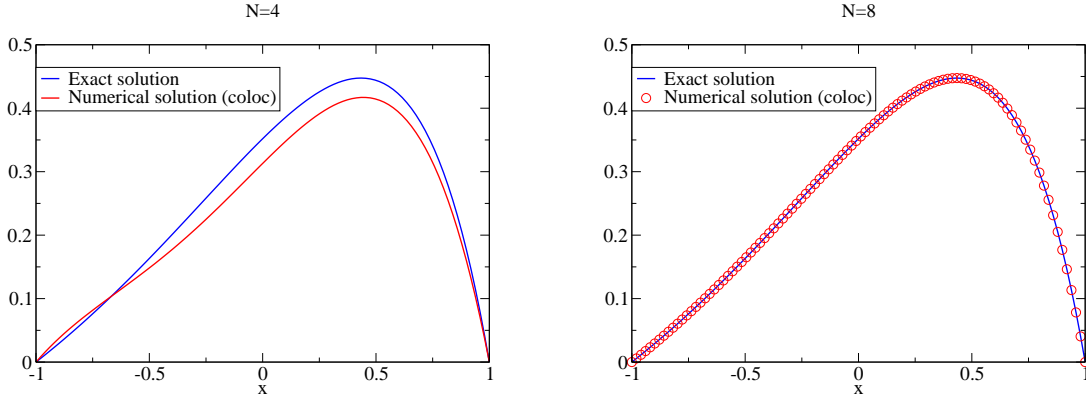


Figure 13: *Same as Fig. 12 but for the collocation method.*

Let us note that, even if the collocation method imposes that  $Lu$  and  $S$  coincide at each collocation point, the unknowns of the system written in the form (62) are the coefficients  $\tilde{u}_n$  and not the  $u(x_n)$ . As for the Tau-method, the system (62) is not invertible and boundary conditions must be enforced by additional equations. In this case, the relaxed conditions are the two associated with the outermost points, i.e.  $n = 0$  and  $n = N$ , which are replaced by appropriate boundary conditions to get an invertible system.

Fig. 13 shows both the exact and numerical solutions for Eq. (58).

#### 2.5.4 Galerkin method

The basic idea of Galerkin method is to seek the solution  $u$  as a sum of polynomials  $G_i$  that *individually* verify the boundary conditions. Doing so  $u$  automatically fulfills those conditions and they do not have to be imposed by additional equations. Such polynomials constitute a Galerkin basis of the problem. For practical reasons, it is better to choose a Galerkin basis that can be expressed easily in terms of the original orthogonal polynomials.

For instance, with the boundary conditions (59), one can choose :

$$G_{2k}(x) = T_{2k+2}(x) - T_0(x) \quad (63)$$

$$G_{2k+1}(x) = T_{2k+3}(x) - T_1(x) \quad (64)$$

More generally, the Galerkin basis relates to the usual ones by means of a transformation matrix

$$G_i = \sum_{j=0}^N M_{ji} T_j, \quad \forall i \leq N-2. \quad (65)$$

Let us mention that the matrix  $M$  is not square. Indeed, to maintain the same degree of approximation, one can consider only  $N-1$  Galerkin polynomials, due to the two additional conditions they have to fulfill (see for instance Eqs. (63-64)). One can also note that, in general, the  $G_i$  are *not* orthogonal polynomials.

The solution  $u$  is sought in terms of the coefficients  $\tilde{u}_i^G$  on the Galerkin basis :

$$u(x) = \sum_{k=0}^{N-2} \tilde{u}_k^G G_k(x). \quad (66)$$

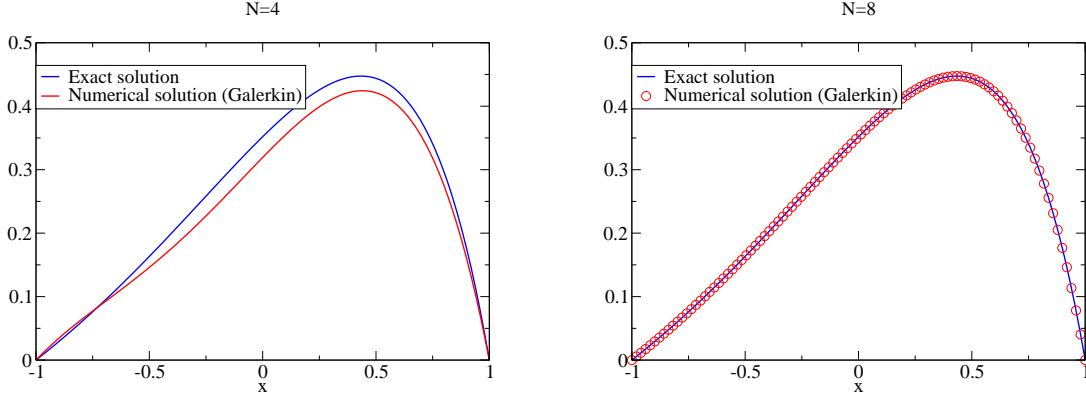


Figure 14: Same as Fig. 12 but for the Galerkin method.

By making use of Eqs. (55) and (65) one can express  $Lu$  in terms of the  $\tilde{u}_i^G$  :

$$Lu(x) = \sum_{k=0}^{N-2} \tilde{u}_k^G \sum_{i=0}^N \sum_{j=0}^N M_{jk} L_{ij} T_i(x). \quad (67)$$

The test functions used in the Galerkin method are the  $G_i$  themselves so that the residual system reads :

$$(Lu, G_n) = (S, G_n), \quad \forall n \leq N-2 \quad (68)$$

where the left-hand-side is computed by means of (67) and by expressing the  $G_i$  in terms of the  $T_i$  by (65). Concerning the right-hand-side, the source itself *is not* expanded on the Galerkin basis, given that it does not fulfill the boundary conditions. Putting all the pieces together, the Galerkin system reads :

$$\sum_{k=0}^{N-2} \tilde{u}_k^G \sum_{i=0}^N \sum_{j=0}^N M_{in} M_{jk} L_{ij} (T_i | T_i) = \sum_{i=0}^N M_{in} \tilde{s}_i (T_i | T_i), \quad \forall n \leq N-2. \quad (69)$$

This is a system of  $N-1$  equations for the  $N-1$  unknowns  $\tilde{u}_i^G$  and it can be directly solved, being well-posed. Once the  $\tilde{u}_i^G$  are known, one can obtain the solution in terms of the usual basis by making, once again, use of the transformation matrix :

$$u(x) = \sum_{i=0}^N \left( \sum_{n=0}^{N-2} M_{in} \tilde{u}_n^G \right) T_i. \quad (70)$$

The solution obtained by the application of this method to the equation (58) is shown on Fig. 14.

### 2.5.5 The methods are optimal

A numerical method is said to be optimal if it does not introduce an additional error to the one that would be done by interpolating the exact solution of a given equation.

Let us call  $u_{\text{exact}}$  such exact solution, unknown in general. Its interpolant is  $I_N u_{\text{exact}}$  and the numerical solution of the equation is  $u_{\text{num}}$ . The numerical method is then optimal if and only if  $\|I_N u_{\text{exact}} - u_{\text{exact}}\|_{\infty}$  and  $\|u_{\text{num}} - u_{\text{exact}}\|_{\infty}$  behave in the same manner when  $N \rightarrow \infty$ .

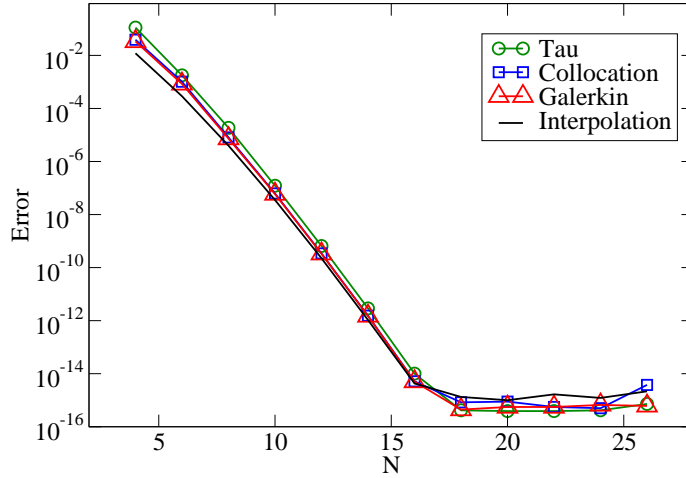


Figure 15: *Difference between the exact solution (60) of Eq. (58) and its interpolant (black curve) and between the exact and numerical solutions for i) the tau method (green curve and circle symbols) ii) the collocation method (blue curve and square symbols) iii) the Galerkin method (red curve and triangle symbols).*

In general, optimality is difficult to check because both  $u_{\text{exact}}$  and its interpolant are unknown. However, for the test problem proposed in Sec. 2.5.2 this can be done. Fig. 15 shows the maximum relative difference between the exact solution (60) and its interpolant and the various numerical solutions. All the curves behave in the same manner as  $N$  increases, indicating that the three methods previously presented are optimal (at least for this particular case).

## 2.6 Multi-domain Techniques

### 2.6.1 Motivations and setting

As seen in Sec. 2.4.3, spectral methods are very efficient when dealing with  $\mathcal{C}^\infty$  functions. However, they lose some of their appeal when dealing with less regular functions, the convergence to the exact functions being substantially slower. Nevertheless, at times, the physicist has to deal with such fields. This is the case for the density jump at the surface of strange stars or the formation of shocks to mention only two examples.

In order to maintain spectral convergence, one then needs to introduce several computational domains such that the various discontinuities of the functions lie at the interface between the domains. Doing so, *in each domain*, one only deals with  $\mathcal{C}^\infty$  functions.

In the following, three different multi-domain methods are presented to solve an equation of the type  $Lu = S$  on  $[-1, 1]$ .  $L$  is a second order linear operator and  $S$  a given source function. Appropriate boundary conditions are given at the boundaries  $x = -1$  and  $x = 1$ .

For simplicity the physical space is split into two domains:

- first domain :  $x \leq 0$  described by  $x_1 = 2x + 1$ ,  $x_1 \in [-1, 1]$ ,
- second domain :  $x \geq 0$  described by  $x_2 = 2x - 1$ ,  $x_2 \in [-1, 1]$ .



If  $x \leq 0$ , a function  $u$  is described by its interpolant in terms of  $x_1$ :  $I_N u(x) = \sum_{i=0}^N \tilde{u}_i^1 T_i(x_1(x))$ . The same thing is true for  $x \geq 0$  with respect to the variable  $x_2$ . Such setting is obviously appropriate to deal with problems where discontinuities occur at  $x = 0$ , that is  $x_1 = 1$  and  $x_2 = -1$ .

### 2.6.2 Multi-domain tau method

As for the standard tau-method (see Sec. 2.5.2) and in each domain, the test functions are the basis polynomials and one writes the associated residual equations. For instance in the domain  $x \leq 0$  one gets:

$$(T_n, R) = 0 \implies \sum_{i=0}^N L_{ni} \tilde{u}_i^1 = \tilde{s}_n^1 \quad \forall n \leq N, \quad (71)$$

the  $\tilde{s}^1$  being the coefficients of the source and  $L_{ij}$  the matrix representation of the operator. As for the one-domain case, one relaxes the last two equations, keeping only  $N - 1$  equations. The same thing is done in the second domain.

Two supplementary equations are enforced to ensure that the boundary conditions are fulfilled. Finally, the operator  $L$  being of second order, one needs to ensure that the solution *and* its first derivative are continuous at the interface  $x = 0$ . This translates as a set of two additional equations involving the coefficients in both domains.

So, one considers

- $N - 1$  residual equations in the first domain,
- $N - 1$  residual equations in the second domain,
- 2 boundary conditions,
- 2 matching conditions,

for a total of  $2N + 2$  equations. The unknowns are the coefficients of  $u$  in both domains (i.e. the  $\tilde{u}_i^1$  and the  $\tilde{u}_i^2$ ), that is  $2N + 2$  unknowns. The system is well posed and admits a unique solution.

### 2.6.3 Method based on the homogeneous solutions

The method exposed here proceeds in two steps. First, particular solutions are computed in each domain. Then, appropriate linear combination with the homogeneous solutions of the operator  $L$  are performed to ensure continuity and impose boundary conditions.

In order to compute particular solutions, one can rely on any of the methods exposed in Sec. 2.5. The boundary conditions at the boundary of each domain can be chosen to be (almost) anything. For instance one can use, in each domain, a collocation method to solve  $Lu = S$ , demanding that the particular solution  $u_{\text{part}}$  is zero at both end of each intervals.

Then, in order to have a solution in the whole space, one needs to add homogeneous solutions to the particular ones. In general, the operator  $L$  is of second order and it admits two independent homogeneous solutions  $g$  and  $h$ , in each domain. Let us note that, in some cases, additional regularity conditions can reduce the number of available homogeneous solutions. The homogeneous solutions can either be computed analytically if the operator  $L$  is simple enough or numerically but one then needs to have a method for solving  $Lu = 0$ .

In each domain, the physical solution is a combination of the particular solution and the homogeneous ones of the type :

$$u = u_{\text{part}} + \alpha g + \beta h, \quad (72)$$

where  $\alpha$  and  $\beta$  are constants that must be determined. In the two domains case, we are left with 4 unknowns. The system they must verify is composed of i) 2 equations for the boundary conditions ii) 2 equations for the matching of  $u$  and its first derivative across the boundary between the two domains. The obtained system is called the matching system and generally admits a unique solution.

#### 2.6.4 Variational method

Contrary to the methods previously presented, the variational one is only applicable with Legendre polynomials. Indeed, the method requires that the measure is  $w(x) = 1$ . It is also useful to extract the second order term of the operator  $L$  and to rewrite it like  $Lu = u'' + H$ ,  $H$  being of first order only.

In each domain, one writes the residual equation explicitly :

$$(\xi, R) = 0 \implies \int \xi u'' dx + \int \xi (Hu) dx = \int \xi S dx. \quad (73)$$

The term involving the second derivative of  $u$  is then integrated by parts :

$$[\xi u'] - \int \xi' u' dx + \int \xi (Hu) dx = \int \xi S dx. \quad (74)$$

The test functions are the same as the ones used for the collocation method, i.e. functions being zero at all but one collocation point:  $\xi_i(x_j) = \delta_{ij}$ . By making use of the Gauss quadratures, the various parts of Eq. (74) can then be expressed as:

$$\int \xi'_n u' dx = \sum_{i=0}^N \xi'_n(x_i) u'(x_i) w_i = \sum_{i=0}^N \sum_{j=0}^N D_{ij} D_{in} w_i u(x_j) \quad (75)$$

$$\int \xi_n (Hu) dx = \sum_{i=0}^N \xi_n(x_i) (Hu)(x_i) w_i = w_n \sum_{i=0}^N H_{ni} u(x_i) \quad (76)$$

$$\int \xi_n S dx = \sum_{i=0}^N \xi_n(x_i) S(x_i) w_i = S(x_n) w_n, \quad (77)$$

where  $D_{ij}$  (resp.  $H_{ij}$ ) represent the action of the derivative (resp. of  $H$ ) in the configuration space

$$g'(x_k) = \sum_{j=0}^N D_{kj} g(x_j) \quad (78)$$

$$(Hg)(x_k) = \sum_{j=0}^N H_{kj} g(x_j). \quad (79)$$

For points *strictly* inside each domain, the integrated term  $[\xi u']$  of Eq. (74) vanishes and one gets equations like:

$$-\sum_{i=0}^N \sum_{j=0}^N D_{ij} D_{in} w_i u(x_j) + w_n \sum_{i=0}^N H_{ni} u(x_i) = S(x_n) w_n. \quad (80)$$

This is a set of  $N - 1$  equations for each domains. In the above form, the unknowns are the  $u(x_i)$ , i.e. the solution is sought in the configuration space.

As usual two additional equations are provided by appropriate boundary conditions at both end of the whole domain. One also gets an additional condition by matching the solution across the boundary between the two domains.

The last equation of the system is the matching of the first derivative of the solution. However, instead of writing it “explicitly”, this is done by making use of the integrated term in Eq. (74) and this is actually the crucial step of the whole method. Applying Eq. (74) to the last point  $x_N$  of the first domain, one gets :

$$u'(x_1 = 1) = \sum_{i=0}^N \sum_{j=0}^N D_{ij} D_{iN} w_i u(x_j) - w_N \sum_{i=0}^N H_{Ni} u(x_i) + S(x_N) w_N. \quad (81)$$

The same thing can be done with the first point of the second domain, to get  $u'(x_2 = -1)$  and the last equation of the system is obtained by demanding that  $u'(x_1 = 1) = u'(x_2 = -1)$  and relates the values of  $u$  in both domains.

Before finishing with the variational method, it may be worthwhile to explain why Legendre polynomials are used. Suppose one wants to work with Chebyshev polynomials instead. The measure is then  $w(x) = \frac{1}{\sqrt{1-x^2}}$ . When one integrates the term containing  $u''$  by part one then gets

$$\int -u'' f w dx = [-u' f w] + \int u' f' w' dx \quad (82)$$

Because the measure is divergent at the boundaries, it is difficult, if not impossible, to isolate the term in  $u'$ . On the other hand this is precisely the term that is needed to impose the appropriate matching of the solution.

### 2.6.5 Merits of the various methods

From a numerical point of view, the method based on an explicit matching using the homogeneous solutions is somewhat different from the two others. Indeed, one has to solve several systems in a row but each one is of the same size than the number of points in one domain. On the contrary, for both the variational and the tau method one has to solve only one system but its size is the same as the number of points in whole space, which can be quite big for settings with many domains. However, those two methods do not require to compute the homogeneous solutions, computation that could be tricky depending on the operators involved and on the number of dimensions. It is also true that the Tau-method is somewhat more difficult to generalized to the higher-dimensional case than the collocation method.

The variational method may seem more difficult to implement and is only applicable with Legendre polynomials, prohibiting the use of any FFT algorithms to compute the coefficients. However, on the mathematical grounds, it is the only method which is demonstrated to be optimal. Moreover, some examples have been found where the others methods are not optimal.

The choice of one method or another thus depend on the particularity of the situation. As for the mono-domain space, for simple tests problems, the results are very similar. Fig. 16 shows the maximum error between the analytical solution and the numerical one for the three different methods. All errors are evanescent and reach machine accuracy with the roughly same number of points.

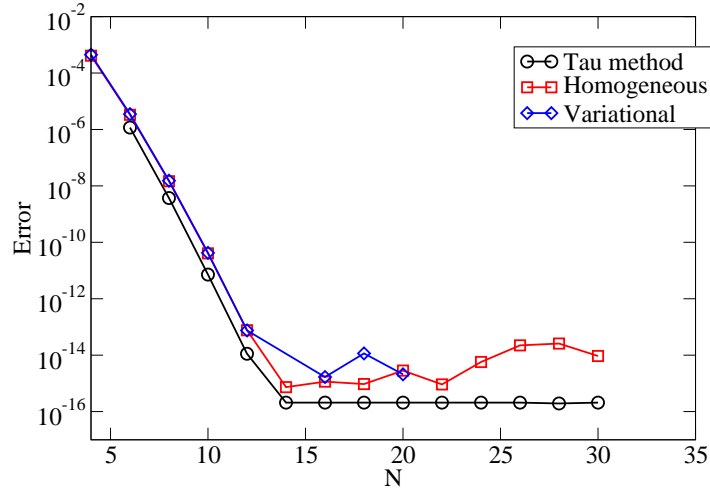


Figure 16: Difference between the exact and numerical solutions of the following test problem.  $\frac{d^2u}{dx^2} + 4u = S$ , with  $S(x < 0) = 1$  and  $S(x > 0) = 0$ . The boundary conditions are  $u(x = -1) = 0$  and  $u(x = 1) = 0$ . The black curve and circles denote results from the multi-domain Tau method, the red curve and triangles from the method based on the homogeneous solutions and the blue curve and diamonds from the variational one.

### 3 Multi-dimensional cases: dealing with space and time

In principle, the generalization to more than one dimension is rather straightforward if one uses the tensorial product. Let us first take an example, with the spectral representation in terms of Chebyshev polynomials of a scalar function  $f(x, y)$ , defined on the square  $(x, y) \in [-1, 1] \times [-1, 1]$ . One simply writes

$$f(x, y) = \sum_{i=0}^M \sum_{j=0}^N a_{ij} T_i(x) T_j(y), \quad (83)$$

with  $T_i$  being the Chebyshev polynomial of degree  $i$ . The partial differential operators can also be generalized, as being linear operators acting on the space  $\mathbb{P}_M \otimes \mathbb{P}_N$ . Simple, linear Partial Differential Equations (PDE) can be solved by one of the methods presented in Section 2.5 (Galerkin, tau or collocation), on this  $MN$ -dimensional space. The development (83) can of course be generalized to any dimension. Some special PDE and spectral base examples, where the differential equation decouples for some of the coordinates, shall be given in Section 3.3. From a relativistic point of view, the time coordinate could be treated in this way and one should be able to achieve spectral accuracy for the time representation of a space-time function  $a(t, x, y, z)$  and its derivatives. Unfortunately, this does not seem to be the case and, we are not aware neither of any efficient algorithm for dealing with the time coordinate, nor of any published successful code solving any of the PDE coming from the Einstein equations.

#### 3.1 Time Discretization

Why is time playing such a special role ? It is not obvious to find in the literature on spectral methods a complete and comprehensive study. A first standard explanation is the difficulty, in general, to predict the exact time interval on which one wants to study the time evolution. Then, time discretization errors in both finite-differences and spectral methods are typically much smaller than are spatial ones. Finally, one must keep in mind that, contrary to finite-differences, spectral methods are storing all global information about a function on the whole time interval. Therefore, a historical reason may be that, since until rather recently there were strong memory and CPU limitations to multi-dimensional simulations, it was not possible to (even hope to) describe a complete field depending on 3+1 coordinates. To this reason one can add the fact that, in the full 3+1 dimensional case, the matrix representing a differential operator would be of very big size; it would therefore be very time-consuming to invert it in a general case, even with iterative methods.

Thus, there have been very few theoretical developments on the subject, with the exception of Ierley *et al.* [86], where the authors have applied spectral methods in time for the study of the Korteweg de Vries and Burger equations, using Fourier series in space and Chebyshev polynomials for the time coordinate. It is interesting to note that they observe a time-stepping restriction: they have to employ multi-domain and patching techniques (see Sec. 2.6) for the time interval, with the size of each sub-domain being roughly given by the Courant-Friedrichs-Lewy (CFL) condition. So the most common approach for time representation are finite-differences techniques, which allow for the use of many well-established time-marching schemes, and the method of lines (for other methods, including fractional stepping, see Fournberg [61]).

##### 3.1.1 Method of lines

Let us write the general form of a first-order in time linear PDE:

$$\frac{\partial u}{\partial t} = Lu, \quad (84)$$

where  $L$  is a linear operator containing only derivatives with respect to spatial coordinates. One can represent the function  $u$  through a finite set  $U_N(t)$ , composed of its time-dependent spectral coefficients, or values at the collocation points. We note  $L_N$  the spectral approximation to the operator  $L$ , together with the boundary conditions, if a tau or collocation method is used.  $L_N$  is therefore represented as an  $N \times N$  matrix. This is the so-called method of lines, which allows one to reduce a PDE to some ODE, after discretization in all dimensions but one. The advantage is that many ODE integration schemes are known (Runge-Kutta, symplectic integrators, ...) and can be used here. We shall suppose an equally-spaced grid in time, with the time-step noted  $\Delta t$  and  $U_N^J = U_N(J \times \Delta t)$ . In order to step from  $U_N^J$  to  $U_N^{J+1}$ , one has either to compute the action of  $L_N$  on  $U_N^J|_{K \leq J}$  (explicit schemes) or to solve for a boundary value problem in term of  $U_N^{J+1}$  (implicit schemes). Both types of schemes have different stability properties, which can be analyzed as follows. Assuming that  $L_N$  can be diagonalized in the sense of the definition given in (3.1.3), the stability study can be reduced to the study of the collection of scalar ODE problems

$$\frac{\partial U_N}{\partial t} = \lambda_i U_N, \quad (85)$$

where  $\lambda_i$  is any of the eigenvalues of  $L_N$  in the sense of Eq. (89).

### 3.1.2 Stability

The basic definition of *stability* for an ODE integration scheme is that, if the time-step is lower than some threshold, then  $\|U_N^J\| \leq A e^{KJ\Delta t}$ , with the constants  $A$  and  $K$  independent of the time-step. This is perhaps not the most appropriate definition, since in practice one often deals with bounded functions and an exponential growth in time would not be acceptable. Therefore, an integration scheme is said to be *absolutely stable* (or asymptotically stable), if  $\|U_N^J\|$  remains bounded,  $\forall J \geq 0$ . This property depends on a particular value of the product  $\lambda_i \times \Delta t$ . For each time integration scheme, the *region of absolute stability* is the set of the complex plane containing all the  $\lambda_i \Delta t$  for which the scheme is absolutely stable.

Finally, a scheme is said to be *A-stable* if its region of absolute stability contains the half complex plane of numbers with negative real part. It is clear that no explicit scheme can be *A-stable* due to the CFL condition. It has been shown by Dahlquist [51] that there is no linear multi-step method of order higher than 2 which is *A-stable*. Thus implicit methods are also limited in time-step size if more than second-order accurate. In addition, Dahlquist [51] shows that the most accurate second-order *A-stable* scheme is the trapezoidal one (also called Crank-Nicolson, or second-order Adams-Moulton scheme)

$$U_N^{J+1} = U_N^J + \frac{\Delta t}{2} (L_N U_N^{J+1} + L_N U_N^J). \quad (86)$$

On Figs. 17 and 18 are displayed the absolute stability regions for the Adams-Bashford and Runge-Kutta families of explicit schemes (see for instance [44]). For a given type of spatial linear operator, the requirement on the time-step usually comes from the largest (in modulus) eigenvalue of the operator. For example, in the case of the advection equation on  $[-1, 1]$ , with a Dirichlet boundary condition

$$\begin{aligned} Lu &= \frac{\partial u}{\partial x}, \\ u(1) &= 0, \end{aligned} \quad (87)$$

and using a Chebyshev-tau method, one has that the largest eigenvalue of  $L_N$  grows in modulus as  $N^2$ . Therefore, for any of the schemes considered on Figs. 17-18, the time-step has a restriction of the type

$$\Delta t \lesssim O(N^{-2}), \quad (88)$$

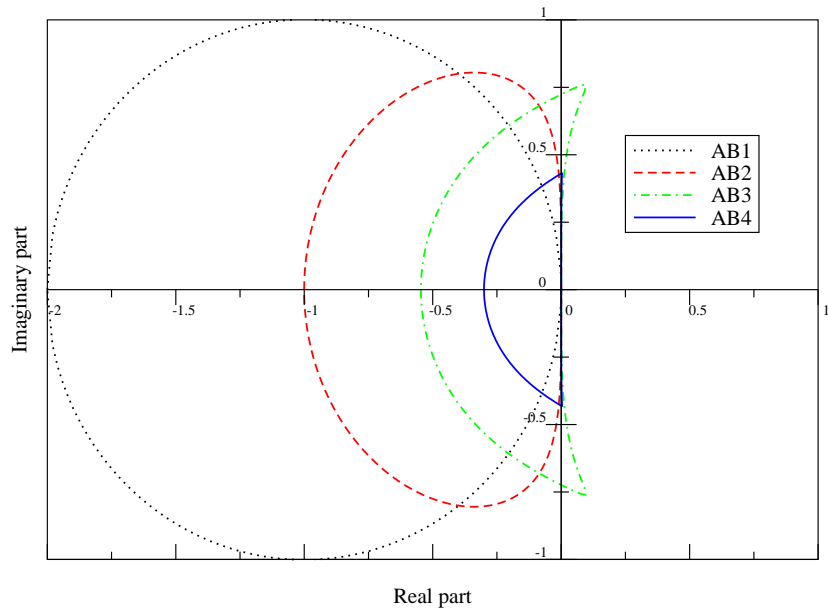


Figure 17: *Regions of absolute stability for the Adams-Bashford integration schemes of order 1 to 4.*

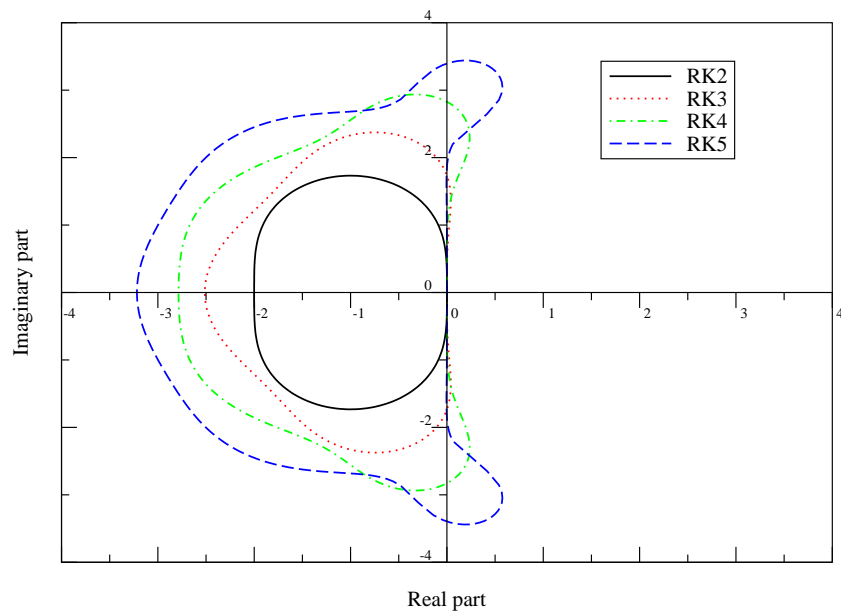


Figure 18: *Regions of absolute stability for the Runge-Kutta integration schemes of order 2 to 5. Note that the size of the region is increasing with the order.*

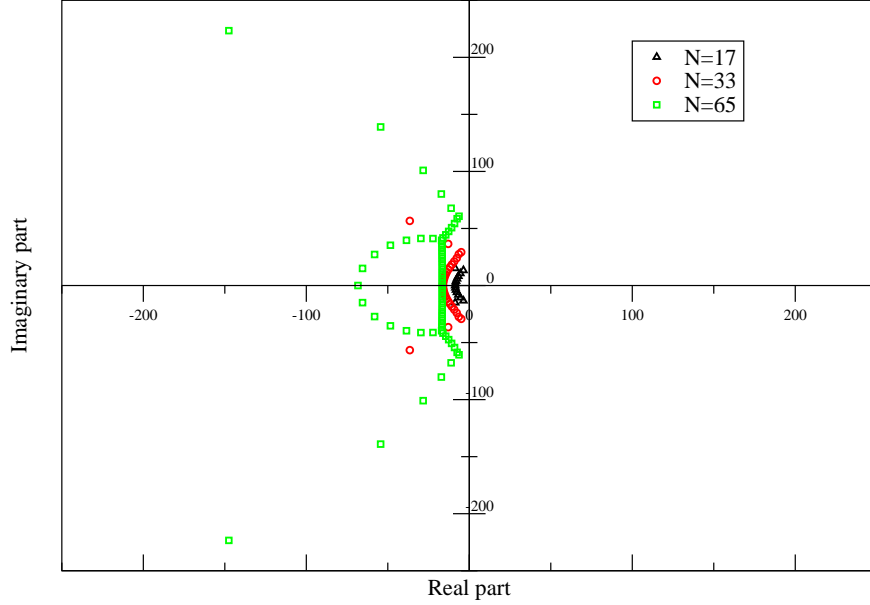


Figure 19: *Eigenvalues of the first derivative-tau operator (89) for Chebyshev polynomials. The largest (in modulus) eigenvalue is not displayed; this one is real, negative and goes as  $O(N^2)$ .*

which can be related to the usual CFL condition by the fact that the minimal distance between two points of a ( $N$ -point) Chebyshev grid decreases like  $O(N^{-2})$ . Due to the above cited *Second Dahlquist barrier* [51], implicit time marching schemes of order higher than two also have such kind of limitation.

### 3.1.3 Spectrum of simple spatial operators

An important issue in determining the absolute stability of a time-marching scheme for the solution of a given PDE is the computation of the spectrum ( $\lambda_i$ ) of the discretized spatial operator  $L_N$  (85). As a matter of fact, these eigenvalues are those of the matrix representation of  $L_N$ , together with the necessary boundary conditions for the problem to be well-posed (*e.g.*  $\mathcal{B}_N u = 0$ ). If one notes  $b$  the number of such boundary conditions, each eigenvalue  $\lambda_i$  (here, in the case of the tau method) is defined by the existence of a non-null set of coefficients  $\{u_j\}_{1 \leq j \leq N}$  such that

$$\begin{aligned} (\forall j) \ 1 \leq j \leq N - b, \quad (L_N u)_j &= \lambda_i u_j, \\ \mathcal{B}_N u &= 0. \end{aligned} \tag{89}$$

As an example, let us consider the case of the advection equation (first-order spatial derivative) with a Dirichlet boundary condition, solved with the Chebyshev-tau method (87). Because of the definition of the problem (89), there are  $N - 1$  “eigenvalues”, which can be computed, after a small transformation, using any standard linear algebra package. For instance, it is possible, making use of the boundary condition, to express the last coefficient as a combination of the other ones

$$u_N = - \sum_{j=1}^{N-1} u_j \tag{90}$$

One is thus left with a usual eigenvalue problem for a  $(N - 1) \times (N - 1)$  matrix. Results are displayed on Figure 19 for three various values of  $N$ . Real parts are all negative: the eigenvalue



which is not displayed lies on the negative part of the real axis and is much larger in modulus (it is growing like  $O(N^2)$ ) than the  $N - 1$  others.

This way of determining the spectrum can be, of course, generalized to any linear spatial operator, for any spectral base, as well as to the collocation and Galerkin methods. Intuitively from CFL-type limitations, one can see that in the case of the heat equation ( $Lu = \partial^2 u / \partial x^2$ ), explicit time-integration schemes (or any scheme which is not  $A$ -stable) shall have a severe time-step limitation of the type

$$\Delta t \lesssim O(N^4), \quad (91)$$

for both Chebyshev or Legendre decomposition bases. Finally, one can decompose a higher-order in time PDE into a first-order system and then use one of the above proposed schemes. In the particular case of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad (92)$$

it is possible to write a second-order Crank-Nicolson scheme directly [110]

$$U_N^{J+1} = 2U_N^J - U_N^{J-1} + \frac{\Delta t^2}{2} \left( \frac{\partial^2 U_N^{J+1}}{\partial x^2} + \frac{\partial^2 U_N^{J-1}}{\partial x^2} \right). \quad (93)$$

Since this scheme is  $A$ -stable, there is no limitation on the time-step  $\Delta t$ , but for explicit or higher-order schemes this limitation would be  $\Delta t \lesssim O(N^2)$ , as for advection equation. The solution of such an implicit scheme is obtained as that of a boundary value problem at each time-step.

### 3.1.4 Semi-implicit schemes

It is sometimes possible to use a combination of implicit and explicit schemes to loosen a time-step restriction of the type (88). Let us consider as an example the advection equation with non-constant velocity on  $[-1, 1]$

$$\frac{\partial u}{\partial t} = v(x) \frac{\partial u}{\partial x}, \quad (94)$$

with the relevant boundary conditions, which shall in general depend on the sign of  $v(x)$ . If on the one hand the stability condition for explicit time schemes (88) is too strong, and on the other hand an implicit scheme is too lengthy to implement or to use (because of the non-constant coefficient  $v(x)$ ), then it is interesting to consider the semi-implicit two-step method (see also [68])

$$\begin{aligned} U_N^{J+1/2} - \frac{\Delta t}{2} L_N^- U_N^{J+1/2} &= U_N^J + \frac{\Delta t}{2} (L_N - L_N^-) U_N^J, \\ U_N^{J+1} - \frac{\Delta t}{2} L_N^+ U_N^{J+1} &= U_N^{J+1/2} + \frac{\Delta t}{2} (L_N - L_N^+) U_N^{J+1/2}, \end{aligned} \quad (95)$$

where  $L_N^+$  and  $L_N^-$  are respectively the spectral approximations to the constant operators  $-v(1)\partial/\partial x$  and  $-v(-1)\partial/\partial x$ , together with the relevant boundary conditions (if any). This scheme is absolutely stable if

$$\Delta t \lesssim \frac{1}{N \max |v(x)|}. \quad (96)$$

With this type of scheme, the propagation of the wave at the boundary of the interval is treated implicitly, whereas the scheme is still explicit in the interior. The implementation of the implicit part, for which one needs to solve a boundary-value problem, is much easier than for the initial operator (94) because of the presence of only constant-coefficient operators. This technique is quite helpful in the case of more severe time-step restrictions (91), for example for a variable coefficient heat equation.

## 3.2 Spatial Coordinate Systems

Most of interesting problems in numerical relativity involve non-symmetric systems and require the use of a full set of three-dimensional coordinates. We briefly review hereafter several coordinate sets (all orthogonal) that have been used in numerical relativity with spectral methods.

- **Cartesian (rectangular) coordinates** are of course the simplest and most straightforward to implement; the line element reads  $ds^2 = dx^2 + dy^2 + dz^2$ . These coordinates are regular in all space, with vanishing connection which makes them easy to use, since all differential operators have simple expressions and the associated triad is also perfectly regular. They are particularly well-adapted to cubic-like domains, see for instance [119, 120], and [62] in the case of toroidal topology.
- **Circular cylindrical coordinates** have a line element  $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$  and exhibit a coordinate singularity on the  $z$ -axis ( $\rho = 0$ ). The associated triad being also singular for  $\rho = 0$ , regular vector or tensor fields have components that are multi-valued (depending on  $\phi$ ) on any point of the  $z$ -axis. As for the spherical coordinates, this can be handled relatively easily with spectral methods. This coordinate system can be useful for axisymmetric or rotating systems, see [10].
- **Spherical (polar) coordinates** will be discussed more in details in Sec. 3.3. Their line element reads  $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$ , showing a coordinate singularity at the origin ( $r = 0$ ) and on the axis for which  $\theta = 0, \pi$ . They are very interesting in numerical relativity for the numerous spherical-like objects under study (stars or black hole horizons) and have been mostly implemented for shell-like domains [32, 82, 119, 148] and for spheres including the origin [35, 82].
- **Prolate spheroidal coordinates** consist of a system of confocal ellipses and hyperbola, describing an  $(x, z)$ -plane, and an angle  $\varphi$  giving the position, as a rotation with respect to the focal axis [95]. The line element is  $ds^2 = a^2 (\sinh^2 \mu + \sin^2 \nu) (d\mu^2 + d\nu^2) + a^2 \sinh^2 \mu \sin^2 \nu d\varphi^2$ . The foci are situated at  $z = \pm a$  and represent coordinate singularities for  $\mu = 0$  and  $\nu = 0, \pi$ . These coordinates have been used in [8] with black hole punctures data at the foci.
- **Bispherical coordinates** are obtained by rotation of bipolar coordinates around the focal axis, with a line element  $ds^2 = a^2 (\cosh \eta - \cos \chi)^{-2} (d\eta^2 + d\chi^2 + \sin^2 \chi d\varphi^2)$ . As for prolate spheroidal coordinates, the foci situated at  $z = \pm a$  ( $\eta \rightarrow \pm\infty, \chi = 0, \pi$ ) and more generally, the focal axis exhibit coordinate singularities. Still, the surfaces of constant  $\eta$  are spheres situated in the  $z > 0 (< 0)$  region for  $\eta > 0 (< 0)$ , respectively. Thus these coordinate are very well adapted for the study of binary systems and in particular for excision treatment of binary black holes [5].

### 3.2.1 Mappings

Choosing a smart set of coordinates is not the end of the story. As for finite-elements, one would like to be able to cover some complicated geometries, like distorted stars, tori, *etc.* or even to be able to cover the whole space. The reason for this last point is that, in numerical relativity, one often deals with isolated systems for which boundary conditions are known only at spatial infinity. A quite simple choice is to perform a mapping from *numerical coordinates* to *physical coordinates*, generalizing the change of coordinates to  $[-1, 1]$ , when using families of orthonormal polynomials or to  $[0, 2\pi]$  for Fourier series.

An example of how to map the  $[-1, 1] \times [-1, 1]$  domain can be taken from Canuto *et al.* [44], and is illustrated on Fig. 20: once known the mappings from the four sides (boundaries) of  $\Omega$

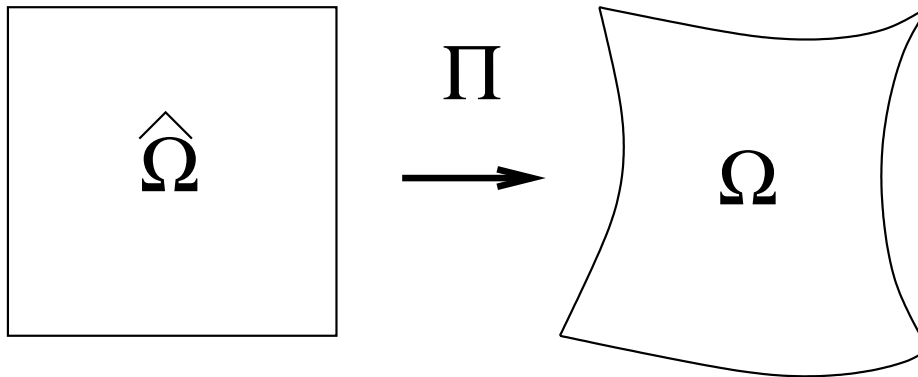


Figure 20: *Regular deformation of the  $[-1, 1] \times [-1, 1]$  square.*

to the four sides of  $\Omega$ , one can construct a two-dimensional regular mapping  $\Pi$ , which preserves orthogonality and simple operators (see Chapter 3.5 of [44]).

The case where the boundaries of the considered domain are not known at the beginning of the computation can also be treated in a spectral way. In the case where this surface corresponds to the surface of a neutron star, two approaches have been used. First in Bonazzola *et al.* [30], the star (and therefore the domain) is supposed to be “star-like”, meaning that there exist a point from which it is possible to reach any point on the surface by straight lines. To such a point is associated the origin of a spherical system of coordinates, so that it is a spherical domain which is regularly deformed to coincide with the shape of the star. This is done within an iterative scheme, at every step once the position of the surface has been determined. The other approach is developed in Ansorg *et al.* [10], using cylindrical coordinates. It is a square in the plane  $(\rho, z)$  which is mapped onto the domain describing the interior of the star. This mapping involves an unknown function, which is itself decomposed on a base of Chebyshev polynomials, so that its coefficients are part of the global vector of unknowns (as the density and gravitational field coefficients).

### 3.2.2 Spatial compactification

As stated above, the mappings can also be used to include spatial infinity into the computational domain. Such a *compactification* technique is not tied to spectral methods and has already been used with finite-differences methods in numerical relativity by *e.g.* Pretorius [122]. However, due to the relatively lower number of degrees of freedom necessary to describe a spatial domain within spectral methods, it is easier within this framework to use some resources to describe spatial infinity and its neighborhood. Many choices are possible to do so, either choosing directly a family of functions well-behaved on an unbounded interval, for example the Hermite functions (see *e.g.* Sec. 17.4 in Boyd [39]), or making use of standard polynomial families, but with an adapted mapping. A first example within numerical relativity was given by Bonazzola *et al.* [33], with the simple inverse mapping in spherical coordinates

$$r = \frac{1}{\alpha(x-1)}, \quad x \in [-1, 1]. \quad (97)$$

This inverse mapping for spherical “shells” has also been used by other authors Kidder & Finn [89], Pfeiffer *et al.* [120, 119], or by Ansorg *et al.* in cylindrical [10] and spheroidal [8] coordinates. Many more elaborated techniques are discussed in Chap. 17 of Boyd [39], but to our knowledge, none has been used in numerical relativity yet. Finally, it is important to point out that, in general, the simple compactification of spatial infinity is not well-adapted to solving hyperbolic

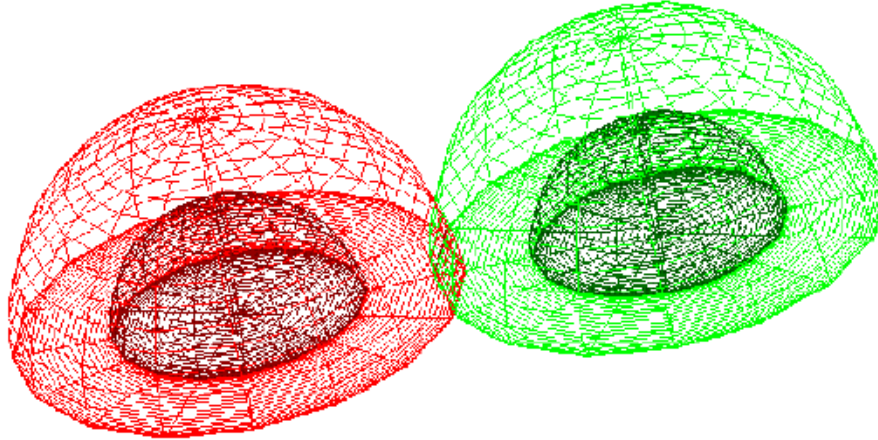


Figure 21: *Two sets of spherical domains describing a binary neutron star or black hole system. Each set is surrounded by a compactified domain of the type (97), which is not displayed*

PDEs and the above mentioned examples were solving only for elliptic equations (initial data, see Sec. 4). For instance, the simple wave equation (92) is not invariant under the mapping (97), as it has been shown *e.g.* by Sommerfeld (see [137], Sec. 23.E). Intuitively, it is easy to see that when compactifying only spatial coordinates for a wave-like equation, the distance between two neighboring grid points becomes larger than the wavelength, which makes the wave poorly resolved after a finite time of propagation on the numerical grid.

### 3.2.3 Patching in more than one dimension

The multi-domain (or multi-patch) technique has been presented in section 2.6 for one spatial dimension. In Bonazzola *et al.* [32] or Grandclément *et al.* [82], the three-dimensional spatial domains consist of spheres (or star-shaped regions) and spherical shells, across which the solution can be matched as in one dimensional problems (only through the radial dependence). In general, when performing a matching in two or three spatial dimensions, the reconstruction of the global solution across all domains might need some more care to clearly write down the matching conditions (see *e.g.* [119], where overlapping as well as non-overlapping domains are used at the same time). For example in two dimension, one of the problems that might arise is the counting of matching conditions for corners of rectangular domains, when such a corner is shared among more than three domains. It is sufficient to impose continuity of either normal derivative at the corner, the jump in the other normal derivative being spectrally small (see Chap. 13 of Canuto *et al.* [44]).

A now typical problem in numerical relativity is the study of binary systems (see also Secs. 4.5 and 5.3) for which two sets of spherical shells have been used by Gourgoulhon *et al.* [74], as displayed on Fig. 21. Different approaches have been proposed by Kidder *et al.* [92], and used by Pfeiffer [119] and Scheel *et al.* [129] where spherical shells and rectangular boxes are combined together to form a grid adapted to binary black hole study. Even more complicated setups to model fluid flows in complicated tubes can be found in [103].

Multiple domains can thus be used to adapt the numerical grid to the interesting part (manifold) of the coordinate space; they can be seen as a technique close to the spectral element method [115]. Moreover, it is also a way to increase spatial resolution in some parts of the computational domain where one expects strong gradients to occur: adding a small domain with many degrees of freedom is the analog of fixed-mesh refinement for finite-differences.

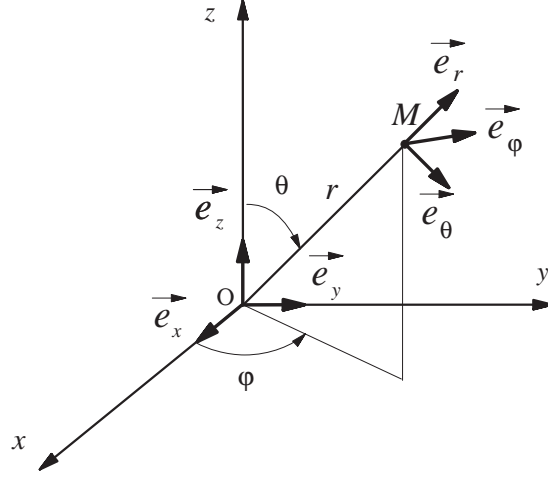


Figure 22: Definition of spherical coordinates  $(r, \theta, \varphi)$  of a point  $M$  and associated triad  $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ , with respect to the Cartesian ones.

### 3.3 Spherical Coordinates and Harmonics

Spherical coordinates (see Fig. 22) are well-adapted for the study of many problems in numerical relativity. Those include the numerical modeling of isolated astrophysical single objects, like a neutron star or a black hole. Indeed, stars' surfaces have spherical-like shape and black hole horizon also have this topology, with is best described in spherical coordinates (eventually through a mapping, see Sec. 3.2.1). As isolated systems in General Relativity, the good boundary conditions are imposed at infinity, requiring a compactification of space, which is here achieved with the compactification of the radial coordinate  $r$  only.

When the numerical grid does not extend to infinity, *e.g.* when solving for a hyperbolic PDE, the boundary defined by  $r = \text{const}$  is a smooth surface, on which boundary conditions are much easier to impose. Finally, *spherical harmonics*, which are strongly linked with these coordinates can simplify a lot the solution of Poisson-like or wave-like equations. On the other hand, there are some technical problems linked with this set of coordinates, as detailed hereafter, but spectral methods can handle them in a very efficient way.

#### 3.3.1 Coordinate singularities

The transformation from spherical  $(r, \theta, \varphi)$  to Cartesian coordinates  $x, y, z$  is obtained by

$$x = r \sin \theta \cos \varphi, \quad (98)$$

$$y = r \sin \theta \sin \varphi, \quad (99)$$

$$z = r \cos \theta. \quad (100)$$

One immediately sees that the origin  $r = x = y = z = 0$  is singular in spherical coordinates because neither  $\theta$  nor  $\varphi$  can be uniquely defined. The same happens for the  $z$ -axis, where  $\theta = 0$  or  $\pi$ , and  $\varphi$  cannot be defined. Among the consequences is the singularity of some usual differential operators, as for instance the Laplace operator

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right). \quad (101)$$

Here, the divisions by  $r$  at the center, or  $\sin \theta$  on the  $z$ -axis look singular. On the other hand, the Laplace operator, expressed in Cartesian coordinates is a perfectly regular one and, if it is applied to a *regular* function, it should give a well-defined result. So the same should be true if one uses spherical coordinates: the operator (101) applied to a regular function should yield a regular result. This means that a regular function of spherical coordinates must have a particular behavior at the origin and on the axis, so that the divisions by  $r$  or  $\sin \theta$  appearing in regular operators are always well-defined. If one considers an analytic function of the (regular) Cartesian coordinates  $f(x, y, z)$ , it can be expanded as a series of powers of  $x, y$  and  $z$ , near the origin

$$f(x, y, z) = \sum_{i,j,k} a_{ijk} x^i y^j z^k. \quad (102)$$

Replacing the coordinate definitions (98)-(100) into this expression gives

$$f(r, \theta, \varphi) = \sum_{n,p,q} a_{npq} r^{n+p+q} \cos^q \theta \sin^{n+p} \theta \cos^n \varphi \sin^p \varphi; \quad (103)$$

and rearranging the terms in  $\varphi$ :

$$f(r, \theta, \varphi) = \sum_{m,p,q} b_{mpq} r^{|m|+2p+q} \sin^{|m|+2p} \theta \cos^q \theta e^{im\varphi}. \quad (104)$$

With some transformations of trigonometric functions in  $\theta$ , one can express the angular part in terms of spherical harmonics  $Y_\ell^m(\theta, \varphi)$ , see Sec. 3.3.2, with  $\ell = |m| + 2p + q$  and obtain the two following regularity conditions, for a given couple  $(\ell, m)$ :

- near  $\theta = 0$ , a regular scalar field is equivalent to  $f(\theta) \sim \sin^{|m|} \theta$ ,
- near  $r = 0$ , a regular scalar field is equivalent to  $f(r) \sim r^\ell$ .

In addition, the  $r$ -dependence translates into a Taylor series near the origin, with the same parity as  $\ell$ . More details in the case of polar (2D) coordinates are given in Chapter 18 of Boyd [39].

If we go back to the evaluation of the Laplace operator (101), it is now clear that the result is always regular, at least for  $\ell \geq 2$  and  $m \geq 2$ . We detail the cases of  $\ell = 0$  and  $\ell = 1$ , using the fact that spherical harmonics are eigenfunctions of the angular part of the Laplace operator (see Eq. (111)). For  $\ell = 0$  the scalar field  $f$  is reduced to a Taylor series of only even powers of  $r$ , therefore the first derivative contains only odd powers and can be safely divided by  $r$ . Once decomposed on spherical harmonics, the angular part of the Laplace operator (101) acting on the  $\ell = 1$  component reads  $-2/r^2$ , which is a problem only for the first term of the Taylor expansion. On the other hand, this term cancels with the  $\frac{2}{r} \frac{\partial}{\partial r}$ , providing a regular result. This is the general behavior of many differential operators in spherical coordinates: when applied to a regular field, the *full* operator gives a regular result, but *single terms* of this operator, may give singular results when computed separately; these singularities canceling between two different terms.

As this may seem an argument against the use of spherical coordinates, let us stress that spectral methods are very powerful in evaluating such operators, keeping everything finite. As an example, we use Chebyshev polynomials in  $\xi$  for the expansion of the field  $f(r = \alpha\xi)$ ,  $\alpha$  being a positive constant. From the recurrence relation on Chebyshev polynomials (42), one has

$$\forall n > 0, \quad \frac{T_{n+1}(\xi)}{\xi} = 2T_n(\xi) - \frac{T_{n-1}(\xi)}{\xi}, \quad (105)$$

which recursively gives the coefficients of

$$g(\xi) = \frac{f(\xi) - f(0)}{\xi} \quad (106)$$

form those of  $f(\xi)$ . The computation of this *finite part*  $g(\xi)$  is always a regular and linear operation on the vector of coefficients. Thus, the singular terms of a regular operator are never computed, but the result is the good one, as if the cancellation of such terms had occurred. Moreover, from the parity conditions it is possible to use only even or odd Chebyshev polynomials, which simplifies the expressions and saves processor time and computer memory. Of course, relations similar to Eq. (105) exist for other families of orthonormal polynomials, as well as relations to divide by  $\sin \theta$  a function developed on a Fourier base. The combination of spectral methods and spherical coordinates is thus a powerful tool for accurately describing regular fields and differential operators in a sphere [35]. It is also true for the solution of two important PDEs in physics: the Poisson and the wave equations, thanks to the use of spherical harmonics.

### 3.3.2 Spherical harmonics

Spherical harmonics are the pure angular functions

$$Y_\ell^m(\theta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\varphi}, \quad (107)$$

where  $\ell \geq 0$  and  $|m| \leq \ell$ .  $P_\ell^m(\cos \theta)$  are the associated Legendre functions defined by

$$P_\ell^m(x) = \frac{(\ell+m)!}{(\ell-m)!} \frac{1}{2^\ell \ell! \sqrt{(1-x^2)^m}} \frac{d^{\ell-m}}{dx^{\ell-m}} (1-x^2)^\ell, \quad (108)$$

for  $m \geq 0$ . The relation

$$P_\ell^{-m}(x) = \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x) \quad (109)$$

gives the associated Legendre functions for negative  $m$ ; note that the normalization factors can vary in the literature. This family of functions have two very important properties. First, they represent an orthogonal set of regular functions defined on the sphere; thus any regular scalar field  $f(\theta, \varphi)$  defined on the sphere can be decomposed on spherical harmonics

$$f(\theta, \varphi) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{m=\ell} f_{\ell m} Y_\ell^m(\theta, \varphi). \quad (110)$$

Since they are regular, they automatically take care of the coordinate singularity on the  $z$ -axis. Then, they are eigenfunctions of the angular part of the Laplace operator (noted here  $\Delta_{\theta\varphi}$ ):

$$\forall(\ell, m) \quad \Delta_{\theta\varphi} Y_\ell^m(\theta, \varphi) := \frac{\partial^2 Y_\ell^m}{\partial \theta^2} + \frac{1}{\tan \theta} \frac{\partial Y_\ell^m}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y_\ell^m}{\partial \varphi^2} = -\ell(\ell+1) Y_\ell^m(\theta, \varphi), \quad (111)$$

the associated eigenvalues being  $-\ell(\ell+1)$ .

The first property makes the description of scalar fields on spheres very easy: spherical harmonics are used as decomposition base within spectral methods, for instance in geophysics or meteorology, and they could be potentially used in numerical relativity, for example for Cauchy-characteristic evolution or matching [154, 14]. It also helps to describe star-like surfaces being defined by  $r = h(\theta, \varphi)$ , as event or apparent horizons [107, 2]. The search for apparent horizon is also made easier: since the function  $h$  verifies a two-dimensional Poisson-like equation, the linear part can be solved directly, just by dividing by  $-\ell(\ell+1)$  in the coefficient space.

The second property makes the Poisson equation

$$\Delta \phi(r, \theta, \varphi) = \sigma(r, \theta, \varphi) \quad (112)$$

very easy to solve. If the source  $\sigma$  and the unknown  $\phi$  are decomposed onto spherical harmonics, the equation transforms into a set of *ordinary* differential equations for the coefficients (see also [82]):

$$\forall(\ell, m) \quad \frac{d^2 \phi_{\ell m}}{dr^2} + \frac{2}{r} \frac{d\phi_{\ell m}}{dr} - \frac{\ell(\ell+1)\phi_{\ell m}}{r^2} = \sigma_{\ell m}. \quad (113)$$

Then, any ODE solver can be used for the radial coordinate: spectral methods of course (see Sec. 2.5), but other ones have been used too (see *e.g.*, Bartnik *et al.* [17, 18]). The same technique can be used to advance in time the wave equation with an implicit scheme and Chebyshev-tau method for the radial coordinate [35, 110].

The use of spherical harmonics decomposition can be regarded as a basic spectral method, as the Fourier decomposition. There are therefore publicly available “spherical harmonics transform” which consist of a Fourier transform in the  $\varphi$ -direction and a successive Fourier and Legendre transform in the  $\theta$ -direction. A rather efficient one is the SpharmonicsKit/S2Kit [106], but writing one’s own functions is also possible [73].

### 3.3.3 Tensor components

All the discussion in Secs. 3.3.1-3.3.2 has been restricted to scalar fields. For vector, or more generally, tensor fields in three spatial dimensions, a vector basis must be specified to express the components. At this point, it is very important to stress out that the choice of the basis is independent from the choice of coordinates. Therefore, the most straightforward and simple choice, even if one is using spherical coordinates, is the Cartesian triad  $(\mathbf{e}_x = \frac{\partial}{\partial x}, \mathbf{e}_y = \frac{\partial}{\partial y}, \mathbf{e}_z = \frac{\partial}{\partial z})$ . With this basis, from a numerical point of view, all tensor components can be regarded as scalars and therefore, a regular tensor can be defined as a tensor field whose components with respect to this Cartesian frame are expandable in powers of  $x, y$  and  $z$  (as in Bardeen and Piran [16]). Manipulation and solution of PDEs for such tensor fields in spherical coordinates are generalization of the techniques for scalar fields, as for instance, for the vector Poisson equation [82], or for the evolution of the unconstrained Einstein system [91].

The use of an *orthonormal spherical basis*  $(\mathbf{e}_r = \frac{\partial}{\partial r}, \mathbf{e}_\theta = \frac{1}{r} \frac{\partial}{\partial \theta}, \mathbf{e}_\varphi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi})$  (see. Fig. 22) requires some more care, as it is outlined hereafter. The interested reader can also find some details in the works by Bonazzola *et al.* [35, 29]. Nevertheless, there are systems in General Relativity where spherical components of tensors can be useful:

- When doing excision for the simulation of black holes, the boundary conditions on the excised sphere for elliptic equations (initial data) may be better formulated for the spherical components of the shift or the 3-metric [50, 78, 87]. In particular, the component normal to the excised surface is easily identified with the radial component.
- Still in 3+1 approach, the extraction of gravitational radiation in the wave zone is made easier if the perturbation to the metric is expressed in spherical components, because the transverse part is then straightforward to obtain [147].

Problems arise because of the singular nature of the basis itself, in addition to the spherical coordinate singularities. The consequences are first that each component is a multi-valued function at the origin  $r = 0$  or on the  $z$ -axis, and then that components of a given tensor are not independent one from another, meaning that one cannot in general specify each component independently or set it to zero, keeping the tensor field regular. As an example, we consider the gradient  $V^i = \nabla^i \phi$  of the scalar field  $\phi = x$ , where  $x$  is the usual first Cartesian coordinate field. This gradient expressed in Cartesian components is a regular vector field  $V^x = 1, \quad V^y = 0, \quad V^z = 0$ . The spherical



components of  $\mathbf{V}$  read

$$\begin{aligned} V^r &= \sin \theta \cos \varphi, \\ V^\theta &= \cos \theta \cos \varphi, \\ V^\varphi &= -\sin \varphi, \end{aligned} \quad (114)$$

which are all three multi-defined at the origin, and the last two on the  $z$ -axis. In addition, if  $V^\theta$  is set to zero, one sees that the resulting vector field is no longer regular: for example the square of its norm is multi-defined, which is not the good property for a scalar field. As for the singularities of spherical coordinates, these difficulties can be properly handled with spectral methods, provided that the decomposition bases are carefully chosen.

The other drawback of spherical components is that usual partial differential operators mix the components. This is due to the non-vanishing connection coefficients associated with the spherical flat metric [29]. For example, the vector Laplace operator ( $\nabla_j \nabla^j V^i$ ) reads

$$\frac{\partial^2 V^r}{\partial r^2} + \frac{2}{r} \frac{\partial V^r}{\partial r} + \frac{1}{r^2} \left( \Delta_{\theta\varphi} V^r - 2V^r - 2\frac{\partial V^\theta}{\partial \theta} - 2\frac{V^\theta}{\tan \theta} - \frac{2}{\sin \theta} \frac{\partial V^\varphi}{\partial \varphi} \right) \quad (115)$$

$$\frac{\partial^2 V^\theta}{\partial r^2} + \frac{2}{r} \frac{\partial V^\theta}{\partial r} + \frac{1}{r^2} \left( \Delta_{\theta\varphi} V^\theta + 2\frac{\partial V^r}{\partial \theta} - \frac{V^\theta}{\sin^2 \theta} - 2\frac{\cos \theta}{\sin^2 \theta} \frac{\partial V^\varphi}{\partial \varphi} \right) \quad (116)$$

$$\frac{\partial^2 V^\varphi}{\partial r^2} + \frac{2}{r} \frac{\partial V^\varphi}{\partial r} + \frac{1}{r^2} \left( \Delta_{\theta\varphi} V^\varphi + \frac{2}{\sin \theta} \frac{\partial V^r}{\partial \varphi} + 2\frac{\cos \theta}{\sin^2 \theta} \frac{\partial V^\theta}{\partial \varphi} - \frac{V^\varphi}{\sin^2 \theta} \right), \quad (117)$$

with  $\Delta_{\theta\varphi}$  defined in Eq. (111). In particular, the  $r$ -component (115) of the operator involves the other two components. This can make the resolution of a vector Poisson equation, which naturally arises in the initial data problem [48] of numerical relativity, technically more complicated and the technique using scalar spherical harmonics (Sec. 3.3.2) is no longer valid. One possibility can be to use vector, and more generally tensor [105, 163, 147, 41] spherical harmonics as decomposition bases. Another technique might be to build from the spherical components regular scalar fields, which can have a similar physical relevance to the problem. In the vector case, one can think of the following expressions

$$\Theta = \nabla_i V^i, \quad \chi = r_i V^i, \quad \mu = r^i \epsilon_{ijk} \nabla^j V^k, \quad (118)$$

where  $\mathbf{r} = r\mathbf{e}_r$  denotes the position vector and  $\epsilon_{ijk}$  the third rank fully antisymmetric tensor. These scalars are the divergence,  $r$ -component and curl of the vector field. The reader can verify that a Poisson equation for  $V^i$  transforms into three equations for these scalars, expandable onto scalar spherical harmonics. The reason why these fields may be more interesting than Cartesian components is that they can have more physical or geometrical meaning.

### 3.4 Going further

The development of spectral methods linked with the problems arising in the field of numerical relativity has always been active and still is now. Among the various directions of research one can foresee, the most interesting might be the improvement of time-integration techniques and the beginning of higher-dimensional studies. In addition to these, it might also be relevant to consider the development of better-adapted mappings and domains, within the spirit of going from pure spectral methods to spectral elements [115, 22].

#### 3.4.1 High-order time schemes

When using spectral methods in time-dependent problems, it is sometimes frustrating to have so accurate numerical techniques for the evaluation of spatial derivatives, and the integration

of elliptic PDEs, whereas the time derivatives, and hyperbolic PDEs, do not benefit from the exponential convergence. Some tentative studies are being undertaken in order to represent also the time interval by spectral methods [6] and, if these techniques can be applied in general three-dimensional simulations, it would really be a great improvement.

Nevertheless, there are other, also more sophisticated and accurate, time-integration techniques that are currently investigated for several stiff PDEs [88], among which Korteweg-de Vries and nonlinear Schrödinger equations [93]. Many such PDEs share the properties of being stiff (very different time-scales/ characteristic frequencies) and combining low-order non-linear terms with higher-order linear terms. Einstein evolution equations can also be written in such a way [29]. Let us consider a PDE

$$\frac{\partial u}{\partial t} = Lu + \mathcal{N}u, \quad (119)$$

with the notations of Sec. 3.1.1 and  $N$  being a nonlinear spatial operator. Following the same notations and within spectral approximation, one recovers

$$\frac{\partial U_N}{\partial t} = L_N U_N + \mathcal{N}_N U_N. \quad (120)$$

We detail hereafter five methods to solve this type of ODEs (see also [88]):

- **Implicit-explicit** techniques use some explicit multi-step scheme to advance the nonlinear part  $\mathcal{N}_N$ , and an implicit one for the linear one.
- **Split-step** are effective when the equation splits into two equation which can be directly integrated (see [93] for examples with the nonlinear Schrödinger and Korteweg-de Vries equations).
- **Integrating factor** is a change of variable that allows for the exact solution of the linear part

$$V_N = e^{-L_N t} U_N, \quad (121)$$

and to use an explicit multi-step method for the integration of the new nonlinear part

$$\frac{\partial V_N}{\partial t} = e^{-L_N t} \mathcal{N}_N e^{L_N t} V_N. \quad (122)$$

- **Sliders** can be seen as an extension of the implicit-explicit method described above. In addition to splitting to problem into a linear and nonlinear part, the linear part itself is split into two or three regions (in Fourier space), depending on the wavenumber. Then, different numerical schemes are used for different groups of wavenumbers: implicit schemes for high wavenumbers and explicit high-order methods for the low wavenumbers. This method is restricted to Fourier spectral techniques in space.
- **Exponential time-differencing** have been known for some time in computational electrodynamics. These methods are similar to the integrating factor technique, but one considers the *exact* equation over one time-step

$$U_N^{J+1} = e^{L_N \Delta t} U_N^J + e^{L_N \Delta t} \int_0^{\Delta t} e^{-L_N \tau} \mathcal{N}_N(U_N(N\Delta t + \tau), N\Delta t + \tau) d\tau. \quad (123)$$

Various orders for these schemes come from the approximation order of the integral. For example Kassam and Trefethen [88] consider a fourth-order Runge-Kutta type approximation to this integral, where the difficulty comes from the accurate computation of functions which suffer from cancellation errors.

### 3.4.2 More than three spatial dimensions

There have been some interest for the numerical study of black holes in higher dimensions: as well with compactified extra-dimensions [138], as in brane world models [136, 96]; recently, some simulations on the head-on collision of two black holes have already been undertaken [156]. With the relatively low number of degrees of freedom per dimension needed, spectral methods should be very efficient in simulations involving four spatial dimensions, or more. We give here starting points to implement 4-dimensional (as needed by *e.g.* brane world models) spatial representation with spectral methods. The simplest approach is to take Cartesian coordinates  $(x, y, z, w)$ , but a generalization of spherical coordinates  $(r, \theta, \varphi, \xi)$  is also possible and necessitates less computational resources. The additional angle  $\xi$  is defined in  $[0, \pi]$ , with the following relations with Cartesian coordinates

$$\begin{aligned} x &= r \sin \theta \cos \varphi \sin \xi, \\ y &= r \sin \theta \sin \varphi \sin \xi, \\ z &= r \cos \theta \sin \xi, \\ w &= r \cos \xi. \end{aligned}$$

The four-dimensional flat Laplace operator appearing in constraint equations [136] reads

$$\Delta_4 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{3}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2 \phi}{\partial \xi^2} + \frac{2}{\tan \xi} \frac{\partial \phi}{\partial \xi} + \frac{1}{\sin^2 \xi} \Delta_{\theta\varphi} \phi \right), \quad (124)$$

where  $\Delta_{\theta\varphi}$  is the two-dimensional angular Laplace operator (111). As in the three-dimensional case, it is convenient to use the eigenfunctions of the angular part, which are here

$$G_k^\ell(\cos \xi) P_\ell^m(\cos \theta) e^{im\varphi}, \quad (125)$$

with  $k, \ell, m$  integers such that  $|m| \leq \ell \leq k$ .  $P_\ell^m(x)$  are the associated Legendre functions defined by Eq. (108).  $G_k^\ell(x)$  are the associated Gegenbauer functions

$$G_k^\ell(\cos \xi) = (\sin^\ell \xi) G_k^{(\ell)}(\cos \xi) \text{ with } G_k^{(\ell)}(x) = \frac{d^\ell G_k(x)}{dx^\ell}, \quad (126)$$

and  $G_k(x)$  being the  $k$ -th Gegenbauer polynomial  $C_k^{(\lambda)}$  with  $\lambda = 1$ . Since the  $G_k$  are also particular case of Jacobi polynomials with  $\alpha = \beta = 1/2$  (see, for example [95]), they fulfill recurrence relations that make them easy to implement as spectral decomposition basis, like the Legendre polynomials. These eigenfunctions are associated with the eigenvalues  $-k(k+2)$ :

$$\Delta_4 (G_k^\ell(\cos \xi) P_\ell^m(\cos \theta) e^{im\varphi}) = -k(k+2) G_k^\ell(\cos \xi) P_\ell^m(\cos \theta) e^{im\varphi}. \quad (127)$$

So as in 3+1 dimensions, after decomposing on such a basis, the Poisson equation turns into a collection of ODEs in the coordinate  $r$ . This type of construction might be generalized to even higher dimensions, with the choice of appropriate type of Jacobi polynomials for every new introduced angular coordinate.

## 4 Stationary computations and initial data

### 4.1 Introduction

In this section, we restrict ourselves to problems where time does not appear explicitly. This is especially the case for systems which are stationary, like neutron stars in rotation or binary systems on circular orbits. The computation of initial data also falls into this class, given that it consists of finding a particular solution of Einstein equations *at a given time* only. Indeed, when using the standard 3+1 decomposition of spacetime, the initial data that are passed to the evolution equation can not be totally arbitrary and must satisfy a set of equations called Einstein's constraint equations. For more details on the initial data problem we refer to the review by G.B. Cook [48].

So in treating the problems considered here, one can forget about the issues specific to time presented in Sec. 3. One of the most straightforward implication is that the codes can be completely spectral (i.e. with respect to the three dimensions of space) and thus one can hope to achieve very good accuracy with moderate computer resources only.

### 4.2 Single compact stars

The computation of the structure of stationary compact stars dates back to 1939 with the famous solution of Tolman-Oppenheimer-Volkoff. During the last years, the need for accurate models has been more pressing especially with the coming online of the gravitational wave detectors which could help to probe the interior of such compact stars. Isolated stars, in rotation, are essentially axisymmetric but some physical effects can induce some symmetry breaking effect that could lead to the emission of gravitational waves. In the following, one will present some computations that aim at including some of those effects, like the spontaneous symmetry breaking, the inclusion of magnetic field, the effect of exotic dense matter, mainly with strange quarks or the influence of an interior composed of two different superfluids. With the exception of the papers by M. Ansorg [9, 10, 11], the results presented here have been computed by the Meudon group.

#### 4.2.1 Formalisms

The first computation of models of relativistic rotating stars in general relativity, by means of spectral methods, is presented in [33]. The equations are solved in spherical coordinates (see Sec. 3.2). Doing so, the fields only depend on the azimuthal angle  $\theta$  and the radius  $r$ . The fields are expanded on spherical harmonics with respect to the angle and with respect to Chebyshev polynomials with respect to  $r$ . The use of spherical harmonics gives a natural way of dealing with coordinate singularity on the  $z$ -axis. In [33] the whole space is divided into two spherical domains, the outer one extending up to infinity by making use of the compactification in  $1/r$  seen in Sec. 3.2.2. With such setting, Einstein equations reduce to a set of four elliptic equations with sources extending up to infinity that are solved using a version of the algorithm based on matching with the homogeneous solutions (presented in Sec. 2.6.3), for each spherical harmonics. The system is closed by giving a description of the matter, being, for instance a polytropic fluid, with or without magnetic field and is solved by iteration.

In the paper [33], a particular emphasize is put on methods to measure the accuracy of the method. For non-rotating stars, the error is found to decrease exponentially, as the number of coefficients increases, i.e. it is evanescent (see Fig. 5 and 6 of [33]). This is expected as, for non-rotating stars, all the fields are  $C^\infty$  (see Sec. 2.4.3). However, for fastly-rotating configurations, the error only decays as a power-law (see Fig. 7 of [33]). This comes from the fact that quantities like the energy density are no longer  $C^\infty$  across the star's surface, thus causing the appearance of a Gibbs-like phenomenon. Nevertheless, the results are in good agreement (to the level of a 0.1%) with those obtained by other numerical methods, as can be seen in [113].

Spectral convergence can be recovered by using surface-adapted coordinates as first done in [28]. A regular mapping 3.2.1 of the numerical coordinates to the physical ones is introduced so that the surface of the star lies at the boundary between two domains. For polytrops with  $\gamma < 2$ , this gets rid of any Gibbs phenomenon and the error is evanescent once again (see Fig. 5 and 6 of [30]). However, for  $\gamma > 2$ , some quantities are still diverging at the surface but the convergence can be made closer and closer to the spectral one by analytically regularizing the density (see Sec. IV of [30]). Doing so, the error decreases as a power-law but the decrease can be made arbitrary fast at the cost of increasing the number of operations and so the computational time.

Up to 2006, the neutron stars were computed using quasi-isotropic coordinates. However, in order to use those configurations as initial data for evolutionary codes, it may be useful to allow for other choices. Among the possible gauges, the Dirac one is one of the most promising [29]. In [99] models of rotating neutron stars, in the Dirac gauge are computed, for both polytropic and realistic equations of state. Compared to the quasi-isotropic coordinates, the use of this new gauge implies to solve one tensorial-like Poisson equation. Configurations obtained with the two different formalisms are shown to be in very good agreement.

#### 4.2.2 Rotating neutron star models

Even before adapting mappings were available, interesting results could be obtained. In two papers [125, 126], models of rotating neutron stars with various equations of state have been computed. Among the most surprising findings, let us mention the existence of supra-massive stars. Those stars do not connect to the non-rotating limit. Indeed, their high mass can only be supported in presence of a centrifugal force. One of the remarkable feature of such stars is the fact that they actually spin up when they lose angular momentum, on the contrary to what is observed for normal stars. This effect can also be seen for neutron stars containing hyperons and thus a softer equation of state [162]. Let us mention that, in this case, the stability analysis of the configurations required the great precision spectral methods with adapted coordinates could provide.

It is known that isolated pulsars spin down due to magnetic breaking. As the rotational frequency decreases, it is possible that the star will encounter a transition from one state of matter to another. Stationary rotating models have been used to determine the properties of such transitions [157, 158]. A puzzling result is that the amount of energy released in a first order phase transition does not depend on the orbital velocity of the star [158] and is the same as for non-rotating ones.

#### 4.2.3 Spontaneous symmetry breaking

It is known that stars can encounter a spontaneous symmetry breaking when rotating fast enough. When such phenomenon occurs, triaxial configurations are formed that are potential emitters of gravitational waves. The departure from axisymmetry is studied in two papers by the Meudon group [28, 27]. The idea of the method is to start from a axisymmetric neutron star configuration and to follow the growth or decay of some triaxial instabilities. Well-established results in the Newtonian results are recovered and this work presents the first results in general relativity, for various equations of states. For a few of them, the frequency at which symmetry-breaking occurs lies in the frequency band of the LIGO and Virgo detectors.

In 2002, this work has been extended in [65] by making use of surface-fitting coordinates. This enables the authors to obtain results in the incompressible case by dealing properly with discontinuities lying at the surface of the star. Classical results in the incompressible case are thus recovered and it is found that the inclusion of relativity has only a moderate effect. Indeed the critical ratio between the kinetic energy and the absolute gravitational one  $T/|W|$  at which the triaxial instability occurs is only 30 % larger for relativistic stars, with respect to their classical counterparts.

If relativistic effects only slightly stabilize the stars, the same is not true for differential rotation. Indeed, in [124], the authors study various rotation profiles and equations of state using the same technique as in [28, 27] to determine the onset of instability. It appears that the critical value of  $T/|W|$  can be almost twice as high as for uniformly rotating stars.

#### 4.2.4 Configurations with magnetic field

Even if magnetic fields are strong in neutron stars, the structure of the objects is not affected until it reaches huge values, of the order of at least  $10^{10}$ T. In [25], models of rapidly rotating stars with poloidal fields are constructed, for various equations of state. The magnetic effects are taken into account consistently by solving the appropriate Maxwell equations, also by means of spectral methods. The maximum mass of highly magnetized neutrons stars is found to be higher from 13 to 29 % than for the non-magnetized stars. The magnetic field induces an additional pressure which can help to support more massive stars, thus explaining this increase.

The presence of a magnetic field can also lead to a deformation of the neutron star. Such deformation could lead to the formation of a triaxial configuration, which would then emit gravitational wave. In [28] the emitted signal is computed. Typically the system radiates at two frequencies :  $\Omega$  and  $2\Omega$  where  $\Omega$  is the angular velocity of the star.

In a more recent work by the Meudon group [112], magnetized configurations have been computed using coordinates matched to the surface of the star, thus making the computation much more accurate. Gyromagnetic ratios of rapidly rotating neutron stars of various equations of state are obtained. The limit of a ratio  $g = 2$ , value for a charged black hole, is never reached.

#### 4.2.5 Strange stars

It is believed that the fundamental state of nuclear matter is not the ordinary matter but rather a plasma of deconfined quarks  $u$ ,  $d$  and  $s$ , called *strange matter*. If this is the case, neutron stars would rather be strange stars. The main difference between those two types of compact stars is that strange ones are somewhat smaller and thus more compact. In particular, they would support higher rotation rates. There is a strong density jump at the surface of a strange star and surface-fitting coordinates are required in order to deal with it.

Fastly rotating models of strange stars are computed in [77, 64]. Due to higher compactness, it is found that strange stars can rotate significantly faster than their neutron star counterparts. The attained  $T/|W|$  can be twice as large. As in the neutron star case, supermassive configurations, that spin-up with angular momentum loss are found. The influence of strange matter on the emission of gravitational waves is studied in [66] where viscosity effects and triaxial instabilities are taken, carefully, into account.

It is believed that millisecond pulsars have been spun up by accreting matter from a companion. However, the details of this mechanism are dependent on the nature of the compact object. In [161], the differences between accretion onto a neutron star and onto a strange star are investigated, using 2D stationary models computed by spectral methods.

#### 4.2.6 Quasi-periodic oscillations

Quasiperiodic oscillations (QPOs) are observed in the kHz regime and are believed to be a signature of matter falling onto a compact object. In the standard picture, the frequency of the QPOs, is the one of the last stable orbit around the compact object. Let us mention that the presence of a last stable orbit around an extended body is not an effect of relativity but can also be seen in the Newtonian regime, as shown in [159].

The precise structure of the accreting object has a great influence on the QPO. In a series of papers [160, 67, 3, 20], comparisons are made between observations and various compact stars models that could account for QPOs.

Using the multi-domain feature of Lorene, strange stars with a crust can also be computed [159], one domain describing the interior of the star and another one the crust. It is shown that the presence of the crust could change the value of the QPO by up to 20 %.

#### 4.2.7 More complex configurations

In this section, objects in more exotic configurations are presented. This is an illustration of both the complexity of the neutron stars physics and the ability of spectral methods to deal with complicated systems.

The observation of the glitches in isolated pulsars is consistent with the presence of a superfluid interior. The simplest model is to consider two fluids, one composed of neutron and the other one of protons and electrons, both being superfluid. However, those two components could have different velocities, in particular different rotation rates. Such configurations are computed in [123]. The multi-domain feature of Lorene is crucial to be able to follow the two different fluids. Among the various results obtained, let us mention the confirmation of the existence of prolate-oblate configurations.

Neutron stars are usually assumed to be at zero-temperature. This approximation is, however, no longer true for newborn neutron stars, just after the supernova. The effect of finite temperature on the neutron star structure is taken into account in [152]. The authors found that the newborn neutron stars was unlikely to undergo the bar mode instability but that secular ones could take place and result in a significant emission of gravitational waves. Another interesting result of [152] is the existence of toroidal-like configurations, which appear for a broad range of parameters and before the mass-shedding limit. In such cases, the surface of the star is highly deformed and surface-fitting coordinates are required.

Apart from the Meudon group, axisymmetric rotating neutron stars have been computed in [9, 10] by a code developed by M. Ansorg and collaborators. The code is based on Lewis-Papapetrou coordinates  $(\rho, \xi)$ , which are closely related to usual cylindrical coordinates. Typically space is decomposed into two domains : one for the interior of the star and another to the exterior which extend up to spatial infinity. Compactification of space and surface-fitting mappings are used. Both variables are expanded on Chebyshev polynomials. Instead of solving the equations harmonics by harmonics and iterate, as is done by the Meudon group, the equations are written with a collocation method (see Sec. 2.5.3) and solved as one single system. The price to pay is that the size of the system is somewhat larger (i.e. in  $m^2$ ,  $m$  being the total number of coefficients for each coordinates) and has to be solved by iteration (typically the Newton-Raphson's method). With those techniques, impressive accuracy is reached. The surface-fitting coordinates are very general and highly distorted configurations can be follow up to the mass-shedding limit.

The coordinates used in [9, 10] are more general than the ones used by the Meudon group, in the sense that they can account for more complicated configurations. Indeed, the code developed in Meudon is limited to star-like configurations whereas, for instance, toroidal ones can be computed by M. Ansorg code. This is especially what is shown in [11] where relativistic axisymmetric toroidal configurations of matter, known as the Dyson rings, are computed. Such rings have been obtained up to the mass-shedding limit. Transition to the limit of an extreme Kerr black hole is also discussed.

### 4.3 Single black holes

Compared to the compact star case, single black holes have not been very much studied. This is probably because the structure of a stationary black hole is somewhat simpler than the one of a

compact star. However, as will be seen, there is still a few things that can be investigated.

Spacetimes containing a single black hole constitute a good benchmark for numerical methods, a lot of results being known analytically. In [89], the authors have implemented a spectral solver and applied it to various test problems. The solver itself is two dimensional and thus applicable only to axisymmetric systems. There is a single domain that consists of the whole space outside a sphere of given radius (i.e. the black hole). Space is compactified by using the standard variable  $1/r$ . The two physical variables  $(r, \theta)$  are mapped onto squares in  $\mathbb{R}^2$  and then expanded on Chebyshev polynomials. The equations are written using a 2-dimensional collocation method (see Sec. 2.5.3) and the resulting system is solved by an iterative algorithm (here Richardson's method with preconditioning). This solver is applied to solve the Einstein's constraint equations for three different systems : i) a single black hole ii) a single black hole with angular momentum iii) a black hole plus Brill waves. In all three cases, spectral convergence is recovered and accuracy of the order of  $10^{-10}$  is reached with 30 points in each dimension.

A black hole is somewhat simpler than a neutron star, in a sense that there is no need for a description of matter (no equation of state for instance). However, in some formalisms, the presence of a black hole is enforced by imposing non-trivial solution on some surfaces (typically spheres). The basic idea is to demand that the surface is a trapped surface. Such surfaces are known to lie inside event horizons and so are consistent with the presence of a black hole. However, in non-stationary cases, the set of equations to be used is not easy to derive. The authors of [87] implemented, numerically, various set of boundary conditions to investigate their properties. Two different and independent spectral codes are used. Both codes are very close to those used in the case of neutron stars, one of them being based on Lorene library [73] (see Sec. 4.2.1) and the other one has been developed by M. Ansorg and shares a lot of features with [9, 10]. Such numerical tools have proved useful in clarifying the properties of some sets of boundary conditions that could be imposed on black hole horizons.

The work [98] tackles the reverse problem : does a given configuration contain a trapped surface and can it be located, at each time step ? Several algorithms have been proposed in the past to find the locus where the expansion of the outgoing light rays vanishes (thus defining the trapped surface). However [98] is the first implementation that uses spectral methods. The code uses the standard Lorene setting i.e. a multi-domain decomposition of space and spherical coordinates (see Sec. 4.2.1 for more details). The horizon finder has been successfully tested on known configurations, like Kerr-Schild black holes. The use of spectral methods makes it both fast and accurate. Even if the code is using only one set of spherical coordinates (hence its presentation in this section), it can be applied to situations with more than one black hole, like the well-known Brill-Lindquist data.

## 4.4 Rings around black holes

The problem of uniformly rings surrounding a black hole can be viewed as an intermediate step between one body, axisymmetric configurations and the two body problem. Indeed, even if one has to deal with two components, the problem is still axisymmetric. In [12], configurations of a black hole surrounded by one uniformly rotating ring of matter are computed in general relativity. The matter is assumed to be a perfect fluid. To solve the equations, space is divided into five computational domains. One of them describes the ring itself, another one the region around the black hole and another extends up to infinity. One of the difficulty is that the surface of the ring is not known a priori and so the domains must be dynamically adapted to its surface. Cylindrical-type coordinates are used and, in each domain, are mapped onto squares of numerical coordinates. The actual mappings depend on the domain and can be found in Sec. IV of [12].

Numerical coordinates are expanded onto Chebyshev polynomials. The system to be solved is obtained by writing Einstein equations in the collocation space and also includes things like



regularity conditions on the axis and appropriate boundary conditions on both the horizon of the black hole and at spatial infinity. As in [9, 10], the system is solved iteratively, using Newton-Raphson's method.

Both Newtonian and relativistic configurations are computed. The ratio between the mass of the black hole and the mass of the ring has been varied from 0 (no black hole) and up to 144. The inner mass shedding of the ring can be obtained. One of the most interesting results is the existence of configurations for which the ratio  $J_c/M_c^2$  between the black hole angular momentum and the square of its mass exceeds one, contrary to what can be achieved for an isolated black hole.

## 4.5 Binary compact stars

### 4.5.1 Formalism

Systems consisting of binary compact objects are known to emit gravitational waves. Due to this emission, no closed orbits can exist and the objects follow a spiral-like trajectory. It implies that such systems have no symmetries that can be taken into account and full time-evolutions should be made. However, when the objects are relatively far apart, the emission of gravitational waves is small and the inspiral slow. In this regime, one can hope to approximate the real trajectory with a sequence of closed orbits. Moreover, the emission of gravitational waves is known to efficiently circularize eccentric orbits so that only circular orbits are usually considered.

So a great deal of work has been devoted to the computation of circular orbits in general relativity. This can be done by demanding that the system admits an helical Killing vector  $\partial_t + \Omega \partial_\varphi$ ,  $\Omega$  being the orbital angular velocity of the system. Roughly speaking, this means that advancing in time is the same thing as turning the system around its axis. Working on the corotating frame, one is left with a time-independent problem.

Additional approximations must be made in order to avoid any diverging quantities. Indeed, when using the helical symmetry, the system has an infinite lifetime and can fill the whole space with gravitational waves, thus causing quantities like the total mass to be infinite. To deal with that, various techniques can be used, the most simple one consists in preventing the appearance of any gravitational waves. This is usually done by demanding that the *spatial metric* is conformally flat. This is not a choice of coordinates but a true approximation that has a priori no reason to be verified. Indeed, even for a single rotating black hole, one can not find coordinates in which the spatial 3-metric is conformally flat so that we do not expect it to be the case for binary systems. However, both comparisons with post-Newtonian results and non-conformally flat results, tend to show that this approximation is relatively good.

Under those assumptions, Einstein equations reduce to a set of five elliptic equations for the lapse, the conformal factor and the shift vector. Those equations encompass both the Hamiltonian and momentum constraint equations and the trace of the evolution equations. To close the system, one must provide a description of the matter. It is commonly admitted that the fluid is irrotational, the viscosity in neutron stars being too small to synchronize the motion of the fluid with the orbital motion. Under such approximation, the motion of the fluid is described by an additional elliptic equation for the potential of the flow. The matter terms entering the equations via the stress-energy tensor can then be computed, once an equation of state is given. A evolutionary sequence can be obtained by varying the separation between the two stars.

### 4.5.2 Numerical procedure

Up to now, only the Meudon group has solved those equations by means of spectral methods. Two sets of domains are used, one centered on each star. Each set consists in spherical-like domains that match the surface of the star and extend up to infinity. Functions are expanded on spherical

harmonics with respect to the angles  $(\theta, \varphi)$  and Chebyshev polynomials with respect to the radial coordinates. Each Poisson equation  $\Delta N = S_N$  is split into two parts  $\Delta N_1 = S_{N_1}$  and  $\Delta N_2 = S_{N_2}$ , such that  $S_N = S_{N_1} + S_{N_2}$  and  $N = N_1 + N_2$ . The splitting is of course not unique and only requires that  $S_{N_i}$  is mainly centered around the star  $i$  so that it is well described by spherical coordinates around it. The equation labelled  $i$  is then solved using the domains centered on the appropriate star. The splittings used for the various equations can be found explicitly in Sec. IV-C of [74].

The elliptic equations are solved using the standard approach by the Meudon group found in [82]. For each spherical harmonics, the equation is solved using a Tau-method and the matching between the various domains is made using the homogeneous method (see Sec. 2.6.3). The whole system of equations is solved by iteration and most of the computational time is spent when quantities are passed from one set of domains to the other one by means of a spectral summation (this requires  $N^6$  operations,  $N$  being the number of collocation points in one dimension). A complete and precise description of the overall procedure can be found in [74].

### 4.5.3 Binary neutron stars

The first sequence of irrotational neutron star binaries computed by spectral means is shown in [32]. Both stars are assumed to be polytropes with an index  $\gamma = 2$ . The results are in good agreement with those obtained, simultaneously, by other groups with other numerical techniques. One of the important point that has been clarified by [32] concerns the evolution of the central density of the stars. Indeed, at the end of the nineties, there was a claim that both stars could collapse, individually, to black holes, before coalescence, due to the increase of central density as the two objects spiral towards each other. Should that have been true, this would have had a great impact on the emitted gravitational wave signal. However it turned out that this was coming from a mistake in the computation of one of the matter term. The correct behavior, confirmed by various groups and in particular by [32], is a decrease in the central density as the stars get closer and closer (see Fig. I of [32]).

If the first sequence computed by spectral methods is shown in [32], the complete description and validation of the method are given in [74]. Convergence of the results with respect to the number of collocation points is exhibited. Global quantities like the total energy or angular momentum are plotted as a function of the separation and show remarkable agreement with results coming from analytical methods (see Figs. 8 to 15 of [74]). Relativistic configurations are also shown to converge to the Newtonian counterparts when the compactness of the stars is small (Figs. 16 to 20 of [74]).

Newtonian configurations of compact stars with various equations of state are computed for both equal masses [145] and various mass ratios [142]. One of the main result of the computations concerns the nature of the end point of the sequence. For equal masses, the sequence ends at contact for synchronized binaries and at mass shedding for irrotational configurations. This is to be contrasted with the non-equal mass case where the sequence always ends at the mass shedding limit of the smallest object.

Properties of the sequences in the relativistic regime are discussed in [143, 144]. In [143] sequences with  $\gamma = 2$  are computed, for various compactness and mass ratios, for both synchronized and irrotational binaries. The nature of the end point of the sequences is discussed and similar behavior than in the Newtonian regime is observed. The existence of a configuration of maximum binding energy is also discussed. Such existence could have some observational implications because it could be an indication of the onset of a dynamical instability. Sequences of polytropes with various indexes ranging from 1.8 to 2.2 are discussed in [144]. In particular, the authors are lead to conjecture that, if a configuration of maximum bounding energy is observed in Newtonian regime, it is also observed in conformal relativity for the same set of parameters.

In [58] the authors derive, from the sequences computed in [143], a method to constraint the compactness of the stars from the observations. Indeed, from results [143], one can easily determine the energy emitted in gravitational waves per interval of frequency (i.e. the power-spectrum of the signal). For large separation, that is for small frequencies, the curves follow the Newtonian one. However, there is a break frequency at the higher end (see Fig. 2 of [58]). The location of this frequency depends mainly on the compactness of the stars. More precisely, the more compact the stars are, the higher the break frequency is. Should such frequency be observed by the gravitational wave detectors, this could help put constraints on the compactness of the neutron stars and thus on the equation of state of such objects.

#### 4.5.4 Extensions

Using the same framework as in [74], sequences of binary compact stars have also been computed [97]. Contrary to the neutron star case, the matter density does not vanish at the surface of the stars and one really needs to use surface-matching domains to avoid any Gibbs phenomenon that would spoil the convergence of the overall procedure. Sequences are computed for both synchronized and irrotational binaries and a configuration of maximum binding energy is attained in both cases. This is not surprising: the strange stars are more compact than the neutron stars and are less likely to be tidally destroyed before reaching the extremum of energy, making it easier to reach dynamical instability.

All the works presented above are done in the conformal flatness approximation. As already stated in Sec. 4.5.1 this is only an approximation and one expects that the true conformal 3-metric will depart from flatness. However, in order to maintain asymptotic flatness of spacetime, one needs to get rid of the gravitational wave content. One such waveless approximation is presented in [133] and implemented in [149]. Two independent codes, one of them being an extension of the work described in [74]. The number of equations to be solved is then greater than in conformal flatness (one has to solve for the conformal metric), but the algorithms are essentially the same. It turned out that the deviation from conformal flatness is rather small. The new configurations are slightly further from post-Newtonian results than the conformally flat ones, which is rather counter-intuitive and might be linked to a difference in the definition of the waveless approximation.

## 4.6 Binary black hole systems

### 4.6.1 Digging the holes

If the computation of binary black holes in circular orbits has a lot of common features with the neutron star case, there is also some differences that need to be addressed. In at least one aspect, black holes are much simpler objects because they are solution of Einstein equations without matter. So the whole issue of investigating various equations of state is irrelevant and there is no need to solve any equation for the matter. However, there is a price to pay and one must find a way to impose the presence of holes in the spacetime. Two main ideas have been proposed and they basically rely on a different point of view on what a black hole is.

In the *puncture method*, the emphasize is put on the central singularity and one demands that various quantities, like the conformal factor, diverge at points called the punctures. Such discontinuities are taken out analytically and the equations are solved numerically for the regular parts, in the whole space. The physical parameters of the holes, like their masses and spins, are encoded in the way the various fields behave at the punctures.

In the *apparent horizon* method, one focuses on the horizons of the holes. By imposing appropriate boundary conditions on, basically, two spheres, one ensures the presence of two black holes. The boundary conditions are based on the concept of trapped surface and apparent horizons. Einstein equations are then solved everywhere *outside* those surfaces. The physical state of the black

holes are precisely encoded in the boundary conditions.

#### 4.6.2 First configurations

The first configurations of binary black holes computed by means of spectral methods can be found in [81]. The formalism and various hypothesis are given in the companion paper [72]. The assumptions are very similar to what is done for binary neutron stars (see Sec. 4.5.1). Helical symmetry is enforced and conformal flatness assumed. The holes are described by the apparent horizon technique. However, the boundary conditions used have been shown to be only approximately valid (up to a rather good accuracy). The numerical techniques are very similar to the ones employed for binary neutron star configurations (see Sec. 4.5.2). Two sets of spherical domains are used, one for each black hole. Boundary conditions are imposed on the surface between the nucleus and the first shell. Both sets extend up to infinity using a compactification in  $1/r$ .

The first application of the helical symmetry to binary black holes can be found in [81] and this technique proved to be valuable. Indeed, for the first time, a good agreement was found between numerical results and post-Newtonian ones. A detailed comparison can be found in [52]. In particular, the location of the minimum of energy is shown to coincide at the level of a few percent. This improvement with respect to previous numerical works is mainly due to difference in the physical hypothesis (i.e. the use of helical symmetry). One important product of [81] is the use of a new criterion to determine the appropriate value of the orbital angular velocity  $\Omega$ . Indeed, for neutron stars, this is done by demanding that the fluid of both stars is in equilibrium [74]. This, of course, is not applicable for black holes. Instead, in [72, 81], it is proposed to find  $\Omega$  by demanding that the ADM mass and the Komar-like mass coincide. Without going into too much details, this amounts to demanding that, far from the binary and at first order in  $1/r$ , the metric behaves like the Schwarzschild one. It is shown in [72] that it can be linked to a relativistic virial theorem. It has since then been shown that this criterion could also be used for neutron stars [144] and that it was equivalent to the use of a variational principle called *the effective potential method* [47].

#### 4.6.3 Further investigations

Apart from the Meudon group, two other spectral codes have been developed in the context of binary black holes and successfully applied to address some of the issues raised by the work of [72, 81].

One of those codes is due to the Cornell group by H. Pfeiffer and collaborators and is described extensively in [120, 119]. The code is multi-domain and two main types of domains are used i) square domains where each Cartesian-like coordinate is expanded on Chebyshev polynomials and ii) spherical domains where spherical harmonics are used for the angles  $(\theta, \varphi)$  and Chebyshev polynomials for the radial coordinate. Space can be compactified by a standard use of the variable  $1/r$ . The two types of domains can be setup in various manner to accommodate the desired geometry. When using both square and spherical domains, there are regions of space that belong to more than one domain. This is to be contrasted with work by the Meudon group where domains are only touching but not overlapping. For overlapping systems one must make the additional requirement that the fields take the same value when expressed in either spectral expansion. The algorithm of [120] solves differential equations by using a multi-dimensional collocation method. The size of resulting system is roughly equal to the number of collocation points. It is then solved iteratively via a Newton-Raphson algorithm with line search. Careful preconditioning is required to ensure convergence of the scheme. Various tests are passed by the code in [120], where elliptic equations and systems are solved in either spherical or bispherical topologies. In the cases presented the error is evanescent.

In [117] the code is used to investigate different ways of solving the constraint equations. Three different decompositions are used: the conformal TT one, the physical TT one and the thin-sandwich decomposition. When solving for the constraint equations only, one also needs to precise some *freely specifiable* variables, which, roughly, describe the physical state of the system. In [117], those specifiable variables are fixed using a superposition of two Kerr-Schild black holes. The net result of [117] is that global quantities, like the total energy, are very sensitive to the choice of decomposition. The variation of total energy can be as large as 5%, which is the order of the energy released by gravitational waves. It is also shown that the choice of extrinsic curvature tensor is more crucial than the one of conformal metric, in concordance with an underlying result of [81]. Let us precise that the equations used in [72, 81] are equivalent to using the thin-sandwich decomposition. The freely specifiable variables are obtained by both the imposition of the helical Killing symmetry and by solving an additional equation for the lapse function (resulting in the so-called *extended thin-sandwich* formalism).

The code developed by the Cornell group has been applied to binary black holes in circular orbits in [47]. This work is an extension of [72, 81]. The difference lies in the use of more elaborate and better boundary conditions on the horizons of the black holes. This has two main applications. First, by allowing for non-vanishing lapse on the horizons, the authors of [47] are solving the constraint equations exactly. This is to be contrasted with results of [81], where the momentum constraint equation was only solved up to some small correction. If this is important conceptually the results from [81] and [47] show a rather good agreement. This is not surprising because the correction used by the Meudon group was known to be small (see Fig. 10 and 11 of [81]). The boundary conditions used in [47] also enable the authors to compute sequences of both corotating and irrotational black holes. Another important result of [47] is the comparison of two criteria for determining the orbital angular velocity  $\Omega$ . They indeed show that the *effective potential method* first introduced in [49] and the method based on the virial theorem proposed in [72] are in very good agreement.

By slightly extending the boundary conditions used in [47], the authors of [116] proposed to reduce to eccentricity of the binary black hole configurations. This is done by giving the holes a small radial velocity by modifying the boundary condition on the shift vector. The code and other equations are the same as in [47]. Time evolution of the obtained initial data show indeed that this technique can reduce the eccentricity of the binary. However, the effect on the emitted gravitational wave is small and probably unimportant.

The other spectral code used to compute configuration of binary black holes is due to M. Ansorg [5]. It shares a lot of features with the code developed by the same author in the context of rotating stars [9, 10] already discussed in Sec. 4.2.7. Space is decomposed in two domains. One of them lies just outside the horizons of the holes and bispherical-like coordinates are used. The other domain extends up to infinity and an appropriate mapping is used in order to avoid the singularity of the bispherical coordinates at spatial infinity (see Sec. IV of [5]). The angle of the bispherical coordinates (i.e. the angle around the x-axis joining the two holes) is expanded onto Fourier series and the two other coordinates onto Chebyshev polynomials. Like in [12, 120], the partial differential equations are solved using a collocation method and the resulting system is solved by iteration, after appropriate preconditioning. The code is used to compute essentially the same configuration as those shown in [47]. An interesting point made in [5] is the detailed investigation of convergence of the results when increasing the resolution. As can be seen in Fig. 4 of [5], the error starts by decreasing exponentially but, for high number of points, it seems that the error only follows a power-law. This is an indication that some Gibbs-like phenomenon must be present. It is conjectured in [5] that this comes from logarithm terms that can not be dealt with properly with a compactification in  $1/r$ . The same kind of effect is investigated in some details in [82], where some criteria for the appearance of such terms are discussed.

A code very similar to the one used in [5] has also been used to compute spacetimes with black

holes using the puncture approach [8]. Given that the black holes are no longer described by their horizons, one does not need to impose inner boundary conditions. The absence of this requirement enabled the author of [8] to use a single domain to describe the whole space, from the puncture up to infinity. The other features of the spectral solver are the same as in [5]. This method has been successfully applied to the computation of binary black hole configurations in the puncture framework. The authors have, in particular, investigated high mass ratios between the bodies and compared their results with the ones given in the test-mass limit around a Schwarzschild black hole. The discrepancy is found to be of the order of 50% for the total energy. It is believed that this comes from the fact that the mass of each puncture can not be directly related to the local black hole mass (see discussion in Sec. VII of [8]).

Let us finally mention that the algorithms developed by M. Ansorg in [9, 10, 8, 5] have all been unified in [7] to accommodate any type of binaries. Various domain decompositions are exhibited that can be used to represent neutron stars, excised black holes or puncture black holes, with compactification of space. The algorithms are shown to be applicable to limiting cases like large mass ratios.

#### 4.7 Black hole-neutron star binaries

Until recently the binaries consisting of a neutron star and a black hole received fewer attention than the other types of systems. It was believed, and this was partly true, that this case could be easily handled once the cases of binary neutron stars and binary black holes were understood. However, such binaries are of evident observational interest and could be the most promising source of gravitational waves for the ground-based detectors [21].

The first application of spectral methods to the black hole-neutron star binaries can be found in [139]. The main approximation is to consider that the black hole is not influenced by the neutron star. Technically, this means that the Einstein equations are split into two parts (i.e. like for binary neutron stars 4.5.2). However the part of the fields associated to the black hole are fixed to their analytical value. As the fields are not solved for the black hole part, the results should depend on the actual splitting, the equations being non-linear. The part of the fields associated with the neutron star are solved using the standard setting for the Meudon group. Of course, this whole procedure is only valid if the black hole is much more massive than the neutron star and this is why [139] is limited to mass ratios of 10. In this particular case, it is shown that the results depend to the level of a few percent on the choice of splitting, which is a measure of the reached accuracy. It is also shown that the state of rotation of the star (i.e. synchronized or irrotational) has few influence on the results.

In [140] the equations of the extended thin-sandwich formulation are solved consistently. Like for the binary neutron star case, two sets of spherical coordinates are used, one centered around each object. The freely specifiable variables are derived from the Kerr-Schild approach. Configurations are obtained with a moderate mass ratio of 5. However the agreement with post-Newtonian results is not very good and the data seem to be rather noisy (especially the deformation of the star). This may be an indication that the Kerr-Schild approach does not lead to accurate quasicircular configurations.

Quasiequilibrium configurations based on an helical Killing vector and conformal flatness have been obtained independently by [80] and [141]. Both codes are based on the Lorene library [73] and use two sets of spherical coordinates. They differ mainly in the choice of boundary conditions for the black hole. However it is shown in the erratum of [80] that the results match pretty well and are in very good agreement with post-Newtonian results. Mass ratios ranging from 1 to 10 are obtained in [141] and the emitted energy spectra are estimated.

## 4.8 Spacetimes with waves

The work [118] presents a method to produce initial data configuration containing waves. Given a fixed background metric, it shows how to superimposed a given gravitational wave content. The equations are solved numerically using a multi-domain spectral code based on [120, 119]. Space is covered by various spherical-like shells and is described up to infinity. Let us mention that the origin is covered by a square domain because regularity conditions at the origin, in spherical coordinates, are not handled by [120, 119]. Such setting is used to generate spacetimes containing i) pure quadrupolar waves ii) flat space with ingoing pulse and iii) a single black hole superimposed with an ingoing quadrupolar wave.

## 4.9 Hyperboloidal initial data

If the 3+1 decomposition is the most widely used for numerical relativity, some other possibilities have been proposed, with possibly better features. In particular, one can vary the foliation of spacetime to get *hyperboloidal data*. With such setting, at infinity, spacetime is foliated by light cones instead of spatial hypersurfaces, which makes extraction of gravitational waves, in principle, easier.

In [62] one is interested in generating hyperboloidal initial data sets from data in physical space. The technique proceeds in two steps. First a non-linear partial differential equation (the Yamabe equation) must be solved to determine the appropriate conformal factor  $\omega$ . Then, the data are constructed by dividing some quantities by this  $\omega$ . This second step involves an additional difficulty:  $\omega$  vanishes at infinity but the ratios are finite and smooth. It is demonstrated in [62] that spectral methods can deal with those two steps. Some symmetry is assumed so that the problem reduces to a 2-dimensional one. The first variable is periodic and expanded on Fourier series whereas Chebyshev polynomials are used for the other one. The Yamabe equation is solved using an iterative scheme based on Richardson's iteration procedure. The construction of the fields, hence the division by a field vanishing at infinity, is then handled by making use of the non-local nature of the spectral expansion (i.e. by working in the coefficient space ; see Sec. 4 of [62] for more details).

## 5 Dynamical Evolution of Relativistic Systems

The modeling of non-stationary physical systems is traditionally the ultimate goal in numerical simulation. Within the field of numerical relativity, the need for studies of dynamical systems is even more pronounced because of the seek for gravitational wave patterns. Unfortunately, as presented in Section 3.1, there is no efficient spectral time discretization yet and one normally uses finite-order time-differentiation schemes. Therefore, the accuracy of time-evolution studies is usually limited by the temporal discretization, and does not exhibit standard spectral (exponential) decay with the number of degrees of freedom. This situation might explain why, except for gravitational collapse [69, 109], very few studies using spectral methods have dealt with dynamical situations until the Cornell/Caltech group began to use spectral methods in numerical relativity, in the beginning of years 2000 [92, 91].

In this section, we review the status of the numerical simulations, using spectral methods, in some fields of General Relativity and Relativistic Astrophysics. Dedicated reviews exist for most of the themes presented here and the interested reader should consult them for physical details and comparisons with other numerical and analytical techniques. Among the systems which have been studied, one can find gravitational collapse [63] (stellar core collapse or collapse of a neutron star to a black hole), oscillations of relativistic stars [94] and evolution of “vacuum” spacetimes. These include the cases of pure gravitational waves or scalar fields, evolving in the vicinity of a black hole or as (self-gravitating) perturbations of Minkowski flat spacetime. Finally, we shall discuss the situation of compact binaries [121, 24] spectral numerical simulations.

### 5.1 Single Stars

The numerical study of the evolution of stars in General Relativity involves two parts: first one has to solve for the evolution of matter (relativistic hydrodynamics, see [60]), and second one must compute the new configuration of the gravitational field. Whereas the second part is now being well studied, in particular since it is the most important to obtain gravitational radiation and to deal with the evolution of black holes (see Sec. 5.2), the first part has not benefited from so many efforts in the past decade. One is facing the paradox: spectral methods have been primarily developed for the simulation of hydrodynamic systems (see Sec. 1.2) but they are not often used for relativistic hydrodynamics. This might be understood as a consequence of the general problem of spectral methods to deal with discontinuous fields: the Gibbs phenomenon (see Sec. 2.4.3). Relativistic flows in astrophysics are often supersonic and therefore contain shocks. Although some techniques have been devised to deal with them in one-dimensional studies (see *e.g.* [36]), there have been no multi-dimensional convincing work. Other problems coming from multi-dimensional relativistic hydrodynamics which can spoil the exponential convergence properties are the density sharp profiles near neutron star surfaces. These can imply a diverging or discontinuous radial derivative of the density, thus slowing down the convergence of the spectral series.

#### 5.1.1 Core Collapse

The physical scenario studied here is the formation of a neutron star from the gravitational collapse of degenerate stellar core. This core can be thought as to be the iron core of a massive star at the end of its evolution (standard mechanism of type II supernova). The degeneracy pressure of the electrons can no longer support the gravity and the collapse occurs. When the central density reaches nuclear values, the strong interaction stiffens the equation of state, stopping the collapse in the central region and a strong shock is generated. This mechanism has been long thought to be a powerful source of gravitational radiation, but recent simulations show that the efficiency is much lower than previously estimated [55, 133]. The appearance of a strong hydrodynamic shock is, in principle, a serious problem to numerical models using spectral methods. Nevertheless,



a first preliminary study in spherical symmetry and in Newtonian theory of gravity has been undertaken in 1986 by Bonazzola and Marck [34], with the use of “natural” viscosity. The authors showed a mass conservation to a level better than  $10^{-4}$  using one domain with only 33 Chebyshev polynomials. In 1993, the same authors performed the first three-dimensional simulation (still in Newtonian theory) of the pre-bounce phase [37], giving a computation of the gravitational wave amplitude, which was shown to be lower than standard estimates. Moreover they showed that, for a given mass, the gravitational wave amplitude depends only on the deformation of the core. These three-dimensional simulations were made possible thanks to the use of spectral methods, particularly for the solution of the Poisson equation for the gravitational potential.

Shock waves give thus difficulties to spectral codes and have been either smoothed with some viscosity, or ignored with the code stopping before their appearance. The idea developed first between the Meudon and Valencia groups was then to use some more appropriate techniques for the simulation of shock waves: namely the High-Resolution Shock-Capturing (HRSC) techniques, also known as Godunov methods (see *Living Reviews* by Martí and Müller [104], and by Font [60]). On the other hand, one wants to keep nice properties of spectral methods, in particular for the solution of elliptic equations or for the representation of more regular fields, like the gravitational one. The combination of both types of methods (HRSC and spectral) was first achieved in 2000 by Novak and Ibáñez [111]. They studied a spherically symmetric core collapse in tensor-scalar theory of gravity, which is a more general theory than General Relativity and allows *a priori* for monopolar gravitational waves. The system of PDEs to be solved resembles the General Relativity one, with the addition of a scalar non-linear wave equation for the monopolar dynamical degree of freedom. It was solved by spectral methods, whereas the relativistic hydrodynamics equations were solved by Godunov techniques. Two grids were used, associated to each numerical technique, and interpolations between both were done at every time-step. Although strong shocks were present in this simulation, they were sharply resolved with HRSC techniques and gravitational field, represented through spectral methods, did not exhibit any Gibbs-like oscillations, and monopolar gravitational waveforms could thus be given. In collaboration with the Garching-hydro group, this numerical technique has been extended in 2005 to three-dimensions, but in the so-called “conformally flat” approximation of General Relativity (see Secs. 4.5 and 4.6) by Dimmelmeyer *et al.* [54]. This approach using spectral methods for the gravitational field computation is now sometimes referred as “Marriage des Maillages” (French for grid wedding) and is currently employed by the core-collapse code CoCoNuT of Dimmelmeyer *et al.* [53, 54] to study general relativistic simulations to a proto-neutron star, with a microphysical equation of state as well as an approximate description of deleptonization [55].

### 5.1.2 Neutron Star Collapse to a Black Hole

To our knowledge, all studies of the neutron star collapse to a black hole, which used spectral methods, are currently restricted to spherical symmetry. However, in this case and contrary to the core-collapse scenario, there is *a priori* no shock wave appearing in the evolution of the system and spectral methods are highly accurate also at modeling the hydrodynamics. Thus, assuming spherical symmetry, the equations giving the gravitational field are very simple, first because of the Birkhoff’s theorem, which gives the gravitational field outside the star, and then from the absence of any dynamical degree of freedom in the gravitational field. For example, when choosing the radial (Schwarzschild) gauge and polar slicing, Einstein equations, expressed within 3+1 formalism, turn into two simple constraints which are simply solved by integrating with respect to the radial coordinate (see [69]).

In this work byourgoulhon, a Chebyshev tau-method is used. The evolution equations for the relativistic fluid variables are integrated with a semi-implicit time scheme and a quasi-Lagrangian grid: the boundary of the grid is comoving with the surface of the star, but the grid points remains

the usual Gauss-Lobatto collocation points (Sec. 2.3.2). Due to the singularity-avoiding gauge choice, the collapsing star ends in the “frozen-star” state, with the collapse of the lapse. This induces strong gradients on the metric potentials, but the code was able to follow the collapse down to very small values of the lapse, at less than  $10^{-6}$ . The code is very accurate at determining whether a star at equilibrium is unstable, by triggering the physical instability from numerical noise at very low level. This property has later been used by Gourgoulhon *et al.* [76] to study the stability of equilibrium configurations of neutron stars near the maximal mass, taking into account the effect of weak interaction processes. The addition of some inward velocity field to initial equilibrium configurations enabled Gourgoulhon [70] to partially answer the question about the minimal mass of black holes: can the effective mass-energy potential barrier associated with stable equilibrium states be penetrated by stars with substantial inward radial kinetic energy? In [70], Gourgoulhon found the possibility to form a black hole with a starting neutron star which was 10% less massive than the usual maximal mass.

The spectral numerical code developed by Gourgoulhon [69] has been extended to also simulate the propagation of neutrinos, coming from thermal effect and non-equilibrium weak interaction processes. With this tool, Gourgoulhon and Haensel [75] have simulated the neutrino bursts coming from the collapse of neutron stars, with different equations of state. Another modification of this spectral code has been done by Novak [109], extending the theory of gravity to tensor-scalar theories. This allowed for the simulation of monopolar gravitational waves coming from the spherically symmetric collapse of a neutron star to a black hole [109]. From a technical point of view, the solution of a non-linear wave equation on curved spacetime has been added to the numerical model. It uses an implicit second-order Crank-Nicolson scheme for the linear terms and an explicit scheme for the non-linear part. In addition, as for the hydrodynamics, the wave equation is solved on a grid, partly comoving with the fluid. The evolution of the scalar field showed that the collapsing neutron star “expelled” all of its scalar charge before the appearance of the black hole.

### 5.1.3 Relativistic Stellar Oscillations

Because of their very high accuracy, spectral methods are able to track dynamical instabilities in the evolution of equilibrium neutron star configurations, as shown in the previous section with the works of Gourgoulhon *et al.* [69, 76]. In these works, when the initial data represented a stable neutron star, some oscillations appeared, which corresponded to the first fundamental mode of the star. As another illustration of the accuracy, let us mention the work by Novak [108], who followed the dynamical evolution of *unstable* neutron stars in tensor-scalar theory of gravity. The instability is linked with the possibility for these stars to undergo some “spontaneous scalarization”, meaning that they could gain a very high scalar charge, whereas the scalar field would be very weak (or even null) outside the star. Thus, for a given number of baryons there would be three equilibria for a star: two stable ones with high scalar charges (opposite in sign) and an unstable one with a weak scalar charge. Starting from this last one, the evolution code described in [109] was able to follow the transition to a stable equilibrium, with several hundreds of damped oscillations for the star. This damping is due to the emission of monopolar gravitational waves, which carry away the star’s kinetic energy. The final state corresponds to the equilibrium configuration, independently computed by a simple code solving the generalized Tolman-Oppenheimer-Volkoff system with a scalar field, up to 1% error, after more than 50000 time-steps.

It is therefore quite natural to try and simulate stellar pulsations (see the *Living Reviews* by Kokkotas and Schmidt [94], and by Anderson and Comer [4]) using spectral methods. Unfortunately, there have been only a few such studies, which are detailed hereafter. The work by Lockitch *et al.* [102] has studied the inertial modes of slowly rotating stars in full general relativity. They wrote down perturbation equations in the form of a system of ordinary differential

equations, thanks to a decomposition onto vector and tensor spherical harmonics. This system is then a nonlinear eigenvalue problem for the dimensionless mode frequency in the rotating frame. Equilibrium and perturbation variables are then expanded onto a basis of Chebyshev polynomials, taking into account the coordinate singularity at the origin and parity requirements. The authors were therefore able to determine the values of the mode frequency making the whole system singular and looked for eigenfunctions, through their spectral decomposition. They found that inertial modes were slightly stabilized by relativistic effects.

A different and maybe more natural approach, namely the time integration of the evolution equations, has been undertaken by Villain *et al.* [150, 151] with a spectral hydrocode, in spherical coordinates. The code solves the linearized Euler or Navier-Stokes equations, with the anelastic approximation. This approximation, which is widely used in other fields of astrophysics and atmospheric physics, consists in neglecting acoustic waves by assuming that time derivatives of the pressure and the density perturbations are negligible. It allows for a characteristic time which is not set by acoustic propagation time, but is much longer and the time-step can be chosen so as to follow the inertial modes themselves. In their 2002 paper [150], the authors study inertial modes (*i.e.* modes whose restoring force is the Coriolis force, among which the  $r$ -modes [4]) in slowly rotating polytrops with  $\gamma = 2$ , in the linear regime. First, this is done in the framework of Newtonian gravity, where the anelastic approximation implies that the Eulerian perturbations of the gravitational potential do not play any role in the velocity perturbations. Second, they study the relativistic case, but with the so-called Cowling approximation, meaning again that the metric perturbations are discarded. In both regimes and trying different boundary conditions for the velocity field at the surface of the star, they note the appearance of a polar part of the mode and the “concentration of the motion” near the surface, showing up in less than 15 periods of the linear  $r$ -mode. A more recent work [151] deals with the study of gravity modes, in addition to inertial modes, in neutron stars. The interesting point of this work is the use of quite a realistic equation of state for nuclear matter, which is valid even when the beta equilibrium is broken. The authors were thus able to show that the coupling between polar and axial modes is increasing with the rotation of the star, and that the coupling of inertial modes with gravity modes in non-barotropic stars can produce fast energy exchanges between polar and axial parts of the fluid motion. From a numerical point of view, one of the key ingredients is the solution of the vector heat equation, coming from the Euler or Navier-Stokes equations. This is done by a poloidal-toroidal [38] decomposition of the velocity field on two scalar potentials, which is very natural within spectral methods. Moreover, to ensure the correct analytical behavior at the origin, all scalar quantities are projected at each time-step to a modified Legendre function basis.

More recently, a complete non-linear study of rotating star pulsations has been set by Dimmelmeier *et al.* [56]. They used the general-relativistic code CoCoNuT (see above, Sec. 5.1.1) in axial symmetry, with a HRSC hydrodynamic solver, and spectral methods for the simplified Einstein equations (conformally flat three-metric). They noted that the conformal flatness condition did not have much effect on the dynamics, when comparing with the Cowling approximation. Nevertheless, they found that differential rotation was shifting the modes to lower frequencies and they confirmed the existence of the mass-shedding induced damping of pulsations.

## 5.2 Vacuum and Black Hole Evolutions

If one wants to simulate the most interesting astrophysical sources of gravitational radiation, one must have a code able to follow, in a stable manner, gravitational waves themselves on a background spacetime. It has been observed by all numerical relativity groups that the stability of a numerical code, which solves Einstein field equations, does not only depend on the numerical algorithm, but also on the particular formulation of the equation. It is therefore a crucial step to devise such a stable formulation, and more particularly with spectral methods, because they add almost no

numerical dissipation and thus, even the smallest instability is not dissipated away and can grow up to unacceptable level. The situation becomes even more complicated with the setup of an artificial numerical boundary at a finite distance from the source, needing appropriate boundary conditions to control the physical wave content, and possibly to limit the growth of unstable modes. All these points have been extensively studied since 2000 by the Caltech-Cornell groups and their pseudospectral collocation code [89, 91, 128, 127, 101, 85, 90, 100, 40]; they have been followed in 2004 by the Meudon group [29] and in 2006 by Tichy [148].

Next, it is necessary to be able to evolve black holes. Successful simulation of binary black holes have been performed using the so-called black-hole puncture technique [43, 15]. Unfortunately, the dynamical part of Einstein fields are not regular at the puncture points and it seems difficult to regularize them so as to avoid any Gibbs-like phenomenon using spectral methods. Therefore the excision technique is employed, removing part of the coordinate space inside the apparent horizon. There is no need for boundary condition on this new artificial boundary, provided that one uses free-evolution scheme [128], solving only hyperbolic equations. On the other hand, with a constrained scheme, elliptic-type equations are to be solved [29] and, as for initial data (see. Sec. 4.3 and 4.6) boundary conditions must be provided *e.g.* on the apparent horizon, from the dynamical horizon formalism [79].

### 5.2.1 Formulation and boundary conditions

Several formulations have been proposed in the literature for the numerical solution of Einstein equations, using spectral methods. The standard one is the 3+1 (a.k.a. Arnowitt-Deser-Misner - ADM) formalism of general relativity [13, 155] (for a comprehensive introduction, see the lecture notes byourgoulhon [71]), which has been reformulated into the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) [19, 132] for better stability. But first, let us mention an alternative *characteristic* approach based on expanding null hypersurfaces foliated by metric 2-spheres developed by Bartnik [17]. This formalism allows for a simple analysis of the characteristic structure of the equations and uses the standard “edth” ( $\eth$ ) operator on  $S^2$  to express angular derivatives. Therefore, Bartnik and Norton [18] used spin-weighted spherical harmonics (see Sec. 3.3.2) to numerically describe metric fields.

Coming back to the 3+1 formalism, Einstein’s equations split into two subsets of equations. First, the *dynamical* equations specifying the way the gravitational field evolves from one time-slice to the next; then, the *constraint* equations which must be fulfilled on each time-slice. Still, it is well-known that for the Einstein system, as well as for the Maxwell’s equations of electromagnetism, if the constraints are verified on the initial time-slice, then the dynamical equations guarantee that they shall be verified in the future of that time-slice. Unfortunately, when numerically doing such *free* evolution, *i.e.* solving only for the dynamical equations, small violations of the constraints due to round-off errors appear to grow exponentially (for an illustration with spectral methods, see *e.g.* [128, 148]). The opposite strategy is to discard some of the evolution equations, keeping the equations for the two physical dynamical degrees of freedom of the gravitational field, and to solve for the four constraint equations: this is a *constrained* evolution [29].

The advantages of the free evolution schemes are that they usually allow for a writing of the Einstein’s equations in the form of a strongly- or symmetric-hyperbolic system, for which there are many mathematical theorems of existence or well-posedness. In addition, it is possible to analyze such systems in terms of characteristics, which can give very clear and easy-to-implement boundary conditions [90]. Using finite-differences numerical methods, it is also very CPU-time consuming to solve for constraint equations, which are of elliptic type, but this is not the case with spectral methods. On the other hand, constrained evolution schemes have by definition the advantage of not being subject to constraint-violation modes. Besides, the equations describing stationary space-times are usually elliptic and are naturally recovered when taking the steady-state

limit of such schemes. Finally, elliptic PDEs usually do not exhibit instabilities and are known to be well-posed. To be more precise, constrained evolution has been implemented by the Meudon group [29], within the framework of BSSN formulation. Free-evolution schemes have been used by Tichy [148] (with the BSSN formulation) and by the Caltech-Cornell group, which has developed their Kidder-Scheel-Teukolsky (KST) scheme [91] and have later used the Generalized-Harmonic (GH) scheme [100]. The KST scheme is in fact a 12-parameters family of hyperbolic formulations of Einstein’s equations, which can be fine-tuned in order to stabilize the evolution of *e.g.* black hole spacetimes [128].

Even when doing so, constraint-violating modes grow exponentially and basically three ways of controlling their growth have been studied by the Caltech-Cornell group. First, the addition of multiples of the constraints to the evolution system in a way to minimize this growth. The parameters linked with these additional terms are then adjusted to control the evolution of the constraint norm. This was called “active constraint control” by Lindblom *et al.* [101], and tested on a particular representation of the Maxwell equations. Second, the same authors devised constraint preserving boundary conditions, where the idea was to get maximally dissipative boundary conditions on the *constraint evolution* equations [101, 90]. This second option appeared to be more efficient, but still did not completely eliminate the instabilities. Finally, bulk constraint violations cannot be controlled by constraint-preserving boundary conditions alone, so Holst *et al.* [85] derived techniques to project at each time-step the solution of the dynamical equations onto the constraint submanifold of solutions. This method necessitates the solution of a covariant inhomogeneous Helmholtz equation to determine the optimal projection. Nevertheless, the most efficient technique seems to be the use of the GH formulation, which also incorporates multiples of the constraints thus exponentially suppressing bulk constraint violation, together with constraint-preserving boundary conditions [100].

Boundary conditions are not only important for the control of the constraint-violation modes in free evolutions. Because they cannot be imposed at spatial infinity (see Sec. 3.2.2), they must be completely transparent to gravitational waves and prevent any physical wave from entering the computational domain. A first study of interest for numerical relativity has been done by Novak and Bonazzola [110], where gravitational waves are considered in the wave zone, as perturbations of flat spacetime. The specificity of gravitational waves is that they start at the quadrupole level ( $\ell = 2$ ) in terms of spherical harmonics expansion. Standard radiative boundary conditions (known as Sommerfeld boundary conditions [137]) being accurate only for the  $\ell = 0$  component, a generalization of these boundary conditions has been done to include quadrupolar terms [110]. They strongly rely on the spectral decomposition of the radiative field in terms of spherical harmonics and on spherical coordinates. More specific boundary conditions for the Einstein system, in order to control the influx of the radiative part of the Weyl tensor, have been devised by Kidder *et al.* [90] for the KST formulation and adapted to the GH formulation by Lindblom *et al.* [100].

### 5.2.2 Gauges and wave evolution

The final ingredient before performing a numerical simulation of the dynamical Einstein system is the gauge choice. For example, the analytical study of the linearized gravitational wave in vacuum has been done with the harmonic gauge, for which the coordinates  $\{x^\mu\}$  verify the scalar covariant wave equation

$$H_\mu = g_{\mu\nu} \nabla_\sigma \nabla^\sigma x^\nu = 0. \quad (128)$$

This is the definition of the form  $H_\mu$ , where  $g_{\mu\nu}$  is the metric and  $\nabla_\sigma$  the associated covariant derivative. Recent works by the Cornell group used the GH formulation in which the gauge choice is achieved through the specification of  $H_\mu$  as an arbitrary function of  $\{x^\mu\}$  and  $G_{\mu\nu}$ , which can be set for instance to its initial value [129]. Still, it is with the KST formulation, and with lapse and shift set from the analytic values, that Boyle *et al.* [40] have submitted their collocation

pseudospectral code to the so-called “Mexico City tests” [1]. These are a series of basic numerical relativity code tests to verify their the accuracy and stability, including small amplitude linear plane wave, gauge wave and Gowdy spacetime evolutions. These tests have been passed by the Caltech-Cornell code, using Fourier basis for all three Cartesian coordinates, and a fourth-order Runge-Kutta time-stepping scheme. In the particular case of the linear plane wave, they exhibited the proper error behavior, as the square of the wave amplitude, because all non-linear terms are neglected in this test. The authors have also shown that the use of filtering of the spherical harmonics coefficients was very effective in reducing nonlinear aliasing instabilities.

Within the constrained formulation of Einstein’s equations, the Meudon group has introduced a generalization of the Dirac gauge to any type of spatial coordinates [29]. Considering the conformal 3+1 decomposition of Einstein’s equations, the Dirac gauge requires that the conformal 3-metric  $\tilde{\gamma}^{ij}$  (such that  $\det \tilde{\gamma}^{ij} = 1$ ) be divergence-free with respect to the flat 3-metric (defined as the asymptotic structure of the 3-metric and with the associated covariant derivative  $\bar{D}$ )

$$\bar{D}_i \tilde{\gamma}^{ij} = 0. \quad (129)$$

Time coordinate is set by the standard maximal slicing condition. These conditions turn to be *dynamical* gauge conditions: the lapse and the shift are determined through the solution of elliptic PDEs at each time-step. With this setting, Bonazzola *et al.* have studied the propagation of a three-dimensional gravitational wave, *i.e.* the solution of the fully nonlinear Einstein equations in vacuum. Their multidomain spectral code based on the LORENE library [73] was able to follow the wave using spherical coordinates, including the (coordinate) singular origin, and to let it out of the numerical grid with transparent boundary conditions [110]. Evolution was performed with a second-order semi-implicit Crank-Nicolson time scheme, and the elliptic system of constraint equations was solved iteratively. Since only two evolution equations were solved (out of six), the other were used as error indicators and proved the awaited second-order time convergence.

### 5.2.3 Black hole spacetimes

As stated at the beginning of Sec. 5.2, the detailed strategy to perform numerical simulations of black hole spacetimes depends on the chosen formulation. With the characteristic approach, Bartnik and Norton [18] modeled gravitational waves propagating on a black hole spacetime, in spherical coordinates but with a null coordinate  $z = t - r$ . They interestingly combined a spectral decomposition on spin-weighted spherical harmonics for the angular coordinates and an eighth-order scheme using spline convolution to calculate derivatives in the  $r$  or  $z$  direction. Integration in these directions was done with a fourth- or eighth-order Runge-Kutta method. For the spectral part, they had to use Orszag’s 2/3 rule [44] for anti-aliasing. This code achieved  $10^{-5}$  as global accuracy and was able to evolve the black hole spacetime up to  $z = 55M$ . More recently, Tichy has evolved a Schwarzschild black hole in Kerr-Schild coordinates in the BSSN formulation, up to  $t \simeq 100M$  [148]. He used spherical coordinates in a shell-like domain, excising the interior of the black hole. The expansion functions are Chebyshev polynomials for the radial direction, and Fourier series for the angular ones.

Most successful simulations in this domain have been performed by the Caltech-Cornell group, who seem to be able to stably evolve forever not only a Schwarzschild, but also a Kerr black hole perturbed by a gravitational wave pulse [100], using their GH formulation with constraint damping and constraint-preserving boundary conditions. However, several attempts have been reported by this group before that, starting with the spherically symmetric evolution of a Schwarzschild black hole by Kidder *et al.* [92]. Problems had arisen when trying three-dimensional simulations of such physical systems with the new parameterized KST formalism [91]. Using spherical coordinates in a shell-like domain, the authors decomposed the fields (or Cartesian components for tensor fields) on a Chebyshev radial base and scalar spherical harmonics. The integration in time was done using

a fourth-order Runge-Kutta scheme and the gauge variables were assumed to keep their analytical initial values. The evolution was limited by the growth of constraint-violating modes at  $t \sim 1000M$ . With a fine-tuning of the parameters of the KST scheme, Scheel *et al.* [128] have been able to extend the lifetime for the numerical simulations to about  $8000M$ . On the other hand, when studying the behavior of a dynamical scalar field on a fixed Kerr background, Scheel *et al.* [127] managed to get nice results on the late time decay of this scalar field. They had to eliminate high-frequency numerical instabilities, with a filter on the spherical harmonics basis, following again Orszag’s 2/3 rule [44] and truncating the last third of coefficients. It is interesting to note that no filtering was necessary on the radial (Chebyshev) basis functions. Somehow more complicated filtering rule has been applied by Kidder *et al.* [90], when trying to limit the growth of constraint-violation in three-dimensional numerical evolutions of black hole spacetimes, with appropriate boundary conditions. They have set to zero the spherical harmonics terms with  $\ell \geq \ell_{\text{max}} - 3$  in the *tensor* spherical harmonics expansion of the dynamical fields. The stable evolutions reported by Lindblom *et al.* [100] thus might be due to the following three ingredients:

- GH formalism, exponentially suppressing all small short-wavelength constraint violations,
- constraint-preserving boundary conditions,
- filtering of spherical harmonics spectral coefficients.

Perhaps, some of these recipes could be helpful to other groups, even working with different numerical methods, to improve their numerical relativity evolution codes.

### 5.3 Binary Systems

As seen in previous section 5.2, not many groups using spectral methods are able to solve all the three-dimensional Einstein equations in a stable way. When dealing with black holes, the situation is even worse. Therefore, there still has not been any complete success, using spectral methods for such simulations of compact object binary systems, at the moment of writing this review. We only report on three recent partial works on the field, dealing with each type of binary system (neutron stars and/or black holes) and leave space for future studies on this rapidly evolving field. We note, of course, that successful numerical evolutions of such systems have been performed with other numerical methods, by Shibata *et al.* [134, 131] for binary neutron stars, Shibata and Uryū [135] for black hole-neutron star binaries and by Pretorius [122], Campanelli *et al.* [43] and Baker *et al.* [15] for binary black holes.

#### 5.3.1 Binary neutron stars

Numerical simulations of the final stage of inspiral and merger of binary neutron stars has been performed by Faber *et al.* [59], who have used spectral methods in spherical coordinates (based on LORENE library [73]) to solve the Einstein equations in the conformally flat approximation (see Secs. 4 and 5.1.1). The hydrodynamic evolution has been computed using a Lagrangian smoothed particle hydrodynamics (SPH) code. As for the initial conditions, described in Sec. 4.5, the equations for the gravitational field reduce, in the case of the conformally flat approximation, to a set of five non-linear coupled elliptic (Poisson-type) PDEs. The considered fields (lapse, shift and conformal factor) are “split” into two parts, each component being associated to one of the stars in the binary. Although this splitting is not unique, the result obtained is independent from it because the equations with the complete fields are solved iteratively, for each time-step. Boundary conditions are imposed to each solution of the field equations at radial infinity, thanks to a multidomain decomposition and a  $u = 1/r$  compactification in the last domain. The authors used  $\sim 10^5$  SPH particles for each run, with an estimated accuracy level of 1 – 2%. Most of the

CPU time was spent in calculating the values of a quantity known at the spectral representation, at SPH particle positions. Another difficulty has been the determination of the domain boundary containing each neutron star, avoiding any Gibbs phenomenon. Because the conformally flat approximation discards gravitational waves, the dissipative effects of gravitational radiation back reaction were added by hand. The authors used the slow-motion approximation [153] to induce a shrinking of the binary systems, and the gravitational waves were calculated with the lowest-order quadrupole formulas. The code has passed many tests and, in particular, they have evolved several quasi-equilibrium binary configurations without adding the radiation reaction force with resulting orbits that were nearly circular (change in separation lower than 4%). The code was thus able to follow irrotational binary neutron stars, including radiation reaction effects, up to the merger and the formation of a differentially rotating remnant, which is stable against immediate gravitational collapse for reasonably stiff equations of state. All the results agreed pretty well with previous relativistic calculations.

### 5.3.2 Black hole-neutron star binaries

A similar combination of numerical techniques has been used by Faber *et al.* [57] to compute the dynamical evolution of merging black hole-neutron star binaries. In addition to the conformally flat approximation, the authors considered only the case of extremely large mass ratio between the black hole and the neutron star, holding thus the black hole position fixed and restricting the spectral computational grid to a neighborhood of the neutron star. The black hole surrounding metric was thus supposed to keep the form of a Schwarzschild black hole in isotropic coordinates. The neutron star was restricted to low compactness (only a few percents) in order to have systems that disrupt well outside the last stable orbit. The system was considered to be in corotation and, as for binary neutron stars, the gravitational radiation reaction was added by hand. As stated above, the numerical methods used SPH discretization to treat dynamical evolution of matter, and the spectral library LORENE [73] to solve the Einstein field Poisson-like equations in the conformally flat approximation. But here, the spectral domains associated with the neutron star did not extend to radial infinity (no compactified domain) and approximate boundary conditions were imposed, using multipole expansion of the fields. The main reason is that the black hole central singularity could not be well described on the neutron star grid.

The authors have studied the evolution of neutron star-black hole binaries with different polytropic indices for the neutron star matter equation of state, the initial data being obtained as solutions of the conformal thin-sandwich decomposition of Einstein equations. They found that, at least for some systems, the mass transfer from the neutron star to the black hole plays an important role in the dynamics of the system. For most of these systems, the onset of tidal disruption occurred outside the last stable orbit, contrary to what had been previously proposed in analytical models. Moreover, they have not found any evidence for significant shocks within the body of the neutron star. This star possibly expanded during the mass loss, eventually losing mass outward and inward, provided that it was not too far within the last stable orbit. Although the major part of released matter remained bound to the black hole, a significant fraction could be ejected with sufficient velocity to become unbound from the binary system.

### 5.3.3 Binary black holes

Encouraging results concerning binary black holes simulations with spectral methods have been obtained by Scheel *et al.* [129]. They have used two coordinate frames to describe the motion of black holes in the spectral grid. Indeed, when using excision technique (punctures are not regular enough to be well represented by spectral methods), excision boundaries are fixed on the numerical grid. This can cause severe problems when, due to the movement of the black hole, the excision surface can become timelike and the whole evolution problem is ill-posed in the absence of boundary



conditions. So one solution may seem to be the using of comoving coordinates, but the authors report that the GH formulation they use appear to be unstable with this setting. They therefore consider a first system of inertial coordinates (with respect to spatial infinity) to define the tensor components in the triad associated with these coordinates; and a second system of comoving (in some sense) coordinates. In the case of their binary black hole tests [129], they define the comoving coordinates dynamically, with a feedback control system that adjusts the moving coordinate frame to control the location of each apparent horizon center.

The spectral code uses 44 domains of different types (spherical and cylindrical shells, rectangular blocks) to describe the binary system. Most of the numerical strategy to integrate Einstein equations is taken from their tests on the GH formulation by Lindblom *et al.* [100] and have already been detailed in Sec. 5.2.1. the important technical ingredient detailed by Scheel *et al.* [129] is the particular filtering of tensor fields in terms spherical harmonics. The dual-coordinate-frame representation can mix the tensor spherical harmonic components of tensors. So, in their filtering of the highest-order tensor spherical harmonic coefficients, the authors had to take into account this mixing by transforming spatial tensors to a rotating frame tensor spherical harmonic basis before filtering and then transforming back to inertial frame basis. this method allowed them to evolve binary black hole spacetimes for more than four orbits, until  $t \gtrsim 600M_{\text{ADM}}$ . There is therefore no doubt about the future capability of spectral methods to also evolve for binary black hole systems until the final merger, with potentially a much higher accuracy.

## 6 Conclusions

We would like to conclude our overview of spectral methods in numerical relativity by pointing a few items that we feel are particularly interesting.

### 6.1 Strengths and Weaknesses

The main advantage of spectral methods, especially with respect to finite difference ones, is the very rapid convergence of the numerical approximant to the real function. This implies that very good accuracy can usually be reached with only a moderate number of points. This obviously makes the codes both faster and less demanding in memory. Various examples of convergence can be found in Sec. 2. However, this rapid convergence is only achieved for  $C^\infty$  functions. Indeed, when the functions are less continuous, spurious oscillations appear and the convergence only follows a power-law. This is the Gibbs phenomenon (see the extreme case of Fig. 11). Gibbs-like phenomenon are very likely to prevent codes from converging or to make time evolutions unstable. So spectral methods must rely heavily on domain decomposition of space and the domains must be chosen so that the various discontinuities lie at the boundaries. Because of this, spectral methods are usually more difficult to implement than standard finite differences (see for instance the intricate mappings of [7]). The situation is even more complicated when the surfaces of discontinuities are not known in advance or have complicated shapes.

Spectral methods are also very efficient at dealing with problems that are related to coordinate singularities. Indeed, if the basis functions fulfill the regularity requirements, then all the functions will automatically satisfy them. In particular, it makes the use of spherical coordinates much easier than with other methods, as explained in Sec. 3.3.

Another nice feature is the fact that a function can be represented either by its coefficients or its values at the collocation points. Depending on the operation one has to perform, it is easier to work on one representation or the other. When working in the coefficients space, one takes full advantage of the non-locality of the spectral representation. A lot of operations that would be difficult otherwise can then be easily performed, like computing the ratio of two quantities vanishing at the same point (see for instance [62]).

### 6.2 Combination with other Methods

Spectral methods have also demonstrated that they can be a valuable tool when combined with other methods. For instance, when shocks are present, spectral methods alone have trouble dealing with discontinuities at the shock interface. However, this can be efficiently dealt with using Godunov methods. Such a combination has already been successfully applied to the simulation of the oscillations of compact stars in [56] and of core collapse [114].

Spectral methods have also been used in conjunction with a description of the fluid based on SPH (smoothed particle hydrodynamics) in the case of binary neutron stars [59] and for the merger of one neutron star and one black hole [57]. In both cases, the fluid is described by an ensemble of particles on which forces are applied. Such a technique can account for complicated fluid configurations, like the spiral arms that are formed during the merger. Such arms would be tricky to follow by means of spectral methods alone. On the other hand, the equations describing gravity are nicely solved by some spectral solvers.

### 6.3 Future Developments

Finally, we would like to point out a few of the directions of work that could lead to interesting results. Of course, we are not aware of what the other groups have planned for the future.

Appropriate choice of coordinates is evidently important. However, for binary systems, rather few results have been using the natural choice of the bispherical coordinates. So far, variations of such coordinates have only been used by M. Ansorg and collaborators and only in the context of initial data [8, 5, 7]. We believe that applying those coordinates, or similar coordinates, to evolutionary codes could lead to interesting results, in terms of both speed and accuracy.

The application of spectral methods to theories more complicated than general relativity is also something that can be imagined. One of the possible fields of application is the one of branes, where there is an additional dimension to spacetime. The fact that spectral methods are accurate with relatively few degrees of freedom, makes them a good candidate to simulate things with extra-dimensions. The addition of gauge fields is also something that could be studied with spectral methods, to investigate the possibility of “hairy” black holes, for instance. Of course, those are just a few leads on what the future applications of spectral methods to the fields of relativity might be.

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